Pro-finite Lie rings and p-adic Lie algebras

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Abstract

Pro-finite Lie rings are the Lie theoretic analogue of pro-finite groups. Pro-finite group theory has often made use of Lie theoretic techniques. This thesis exploits the relationship between groups and Lie rings in the opposite direction, providing a systematic study of pro-finite Lie rings, with the intention of furthering our understanding of both Lie rings and groups.

In particular this thesis provides a detailed study of the class of Lie $\mathbb{Z}_p$-algebras we call pro-p Lie rings (the Lie theoretic analogue to pro-p groups), studies the relationship between the algebraic structure and topology of pro-finite Lie rings, and provides a positive solution to the Kurosh problem for pro-finite Lie rings.

Keywords: Lie ring, p-adic Lie algebra, pro-finite group, pro-p group, p-adic analytic group.

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Chapter 1

Introduction

The theory of pro-finite groups has become important in a variety of mathematical fields. Pro-finite groups arise in number theory as Galois groups of finite field extensions. In analysis pro-finite groups appear in the form of quotient groups of compact Hausdorff topological groups modulo the connected component of the identity. In the form of pro-p groups they have become essential in the attempt to classify finite p-groups by co-class.

However, much of the recently developed theory of pro-p groups relies on Lie-theoretic techniques. For example, Lazard's characterisation of p-adic analytic groups uses the fact that a uniform pro-p group has, in a natural way, the structure of a finitely generated p-adic Lie algebra ([8]). Also, one of the major results making use of pro-p group theory, Zelmanov's celebrated solution of the restricted Burnside problem, is essentially the application of a deep generalisation of Engel's theorem to a certain graded Lie algebra constructed from a finitely generated pro-p group of finite exponent ([31]). A survey of Lie theoretic methods in pro-p groups theory can be found in [24]. The results contained in this thesis seek to further exploit this close relationship between groups and Lie rings, but in the other direction. It is hoped that a systematic study of the Lie theoretic counterpart to pro-finite groups, namely pro-finite Lie rings, will further our understanding of both Lie rings and groups.

In this chapter we will recall the notions of pro-finite and pro-p groups, and introduce the notions of pro-finite and pro-$\mathscr{C}$ Lie rings. Some preliminary results regarding pro-finite and pro-$\mathscr{C}$ Lie rings will also be given. Chapter 2 will discuss pro-p Lie rings, a natural Lie theoretic analogue to pro-p groups, and make a detailed study of their properties including subring growth, p-adic module structure, and Prüfer rank. Most importantly, we deduce, from a deep theorem of Zelmanov, that a finitely generated pro-p Lie ring L is finitely generated as a $\mathbb{Z}_p$-module if and only if, for sufficiently large n, $W_n = \langle e_{12}, te_{22} \rangle \subseteq \mathfrak{gl}_2(\mathbb{F}_p[t]/\langle t^n \rangle)$ is not a closed section of L. In Chapter 3 we extend a result from Chapter 2, demonstrating the uniqueness of the topology of finitely generated pro-nilpotent Lie rings. In Chapter 4 we make use of the pro-p Lie ring theory of Chapter 2 to give a solution to the Kurosh Problem for pro-finite Lie rings and pro-finite restricted Lie algebras.

We refer the reader to [26] for a general discussion of compact topological algebra, [29] and [16] for a rigorous introduction to pro-finite groups, and to the monograph [4] for a comprehensive overview of the relevant pro-p group theory.

1.1 Notation

The following notation is used throughout this thesis. The (topological) closure of a subset X of some topological space is denoted $\overline{X}$. We shall use $A \leq B$ to denote that A is either a subgroup, submodule, or subring (according to context) of B, with $A < B$ denoting proper subgroups, submodules, and subrings. The notation $A \lhd B$ will denote that A is a normal subgroup or ideal of B, again according to context. Closed and open subgroups, submodules, and subrings will be denoted $A \leq_c B$ and $A \leq_o B$ respectively, with closed and open normal subgroups and ideals denoted $A \lhd_c B$ and $A \lhd_o B$ as expected.

For a Lie ring L we shall denote the Lie product by $[x, y]$, with Lie brackets left normed so that $[x, y, z] = [[x, y], z]$, and with the usual notational convention that $[x,_n y] = [[x,_{n-1} y], y]$. We shall use $\gamma_n(L) = [\gamma_{n-1}(L), L] = [L,_{n-1} L]$ to denote the n^{th} term of the lower central series of L. We often write L' instead of $\gamma_2(L)$ for brevity. We shall say that L is (topologically) *finitely generated* if there is a finite subset X of L such that $L = \overline{\langle X \rangle}$. We denote by $d(L)$ the cardinality of a minimal such generating set. We shall use analogous notation for pro-finite groups whenever the need arises.

Finally, for a prime p, $\mathbb{Z}_p$ will denote the ring of p-adic integers and $\mathbb{Q}_p$ its field of fractions, and $\mathbb{F}_p$ will denote the field of p elements.

1.2 Pro-finite and pro-$\mathscr{C}$ Lie rings

As defined in [4], a *pro-finite group* is a compact Hausdorff topological group whose open subgroups form a neighbourhood base of the identity. We define *pro-finite Lie rings* analogously as compact Hausdorff topological Lie rings whose open ideals of finite index form a neighbourhood base of 0. Since each ideal of a pro-finite Lie ring L is a normal subgroup of $(L, +)$, it follows that $(L, +)$ is a pro-finite group. A number of basic properties of pro-finite Lie rings may be deduced as immediate corollaries of this fact:

Lemma 1.2.1. *Let L be a pro-finite Lie ring. Then*

(i) *every open subring of L is closed and of finite index in L;*

(ii) *a closed subring of L is open if and only if it is of finite index in L;*

(iii) *if X and Y are closed subsets of L, and n is an integer, then the subsets* $X + Y = \{x + y | x \in X, y \in Y\}$ *and* $nX = \{nx | x \in X\}$ *are also closed.*

All of which follow from corresponding results for pro-finite groups (see [4]).

A further result regarding pro-finite groups is that given a pro-finite group G and subset $X \subseteq G$ then

$$\overline{X} = \bigcap_{H \lhd_O G} XH,$$

where $XH = \{xH | x \in X\}$ is a union of cosets. We can deduce a corresponding result for pro-finite Lie rings. Let L be a pro-finite Lie ring. Then, since the ideals of L form a neighbourhood base of 0, given the open normal subgroup $H \lhd_O (L, +)$ we can find an ideal $I \lhd_O L$ such that $I \subseteq H$. Thus

$$\overline{X} = \bigcap_{I \lhd_O L} X + I,$$

for any subset $X \subseteq L$.

As already mentioned, a class of pro-finite groups of interest is the class of pro-p groups. A *pro-p group* is a compact Hausdorff topological group G such that the open normal subgroups N, with G/N a finite p-group, form a neighbourhood base of 0. Such groups play a role in pro-finite group theory similar to that of finite p-groups in finite group theory. Before defining our Lie theoretic analogue of pro-p groups we will consider the more general notion of pro-$\mathscr{C}$ Lie rings.

Given a class $\mathscr{C}$ of finite Lie rings, a *pro-$\mathscr{C}$ Lie ring* is a compact Hausdorff topological Lie ring such that the ideals $I \lhd_o L$, with L/I in class $\mathscr{C}$, form a neighbourhood base of 0. Clearly a pro-$\mathscr{C}$ Lie ring is necessarily a pro-finite Lie ring. Furthermore, following directly from the definition is the following:

Proposition 1.2.2. *Let $\mathscr{C}$ be a class of finite Lie rings that is closed under subrings, quotients, and direct sums, and let L be a pro-$\mathscr{C}$ Lie ring. Then*

(i) *if K is a closed subring of L then K is also a pro-$\mathscr{C}$ Lie ring;*

(ii) *if I is a closed ideal of L then L/I, under the quotient topology, is also a pro-$\mathscr{C}$ Lie ring.*

We can also provide an alternative characterisation of pro-$\mathscr{C}$ Lie rings in terms of inverse limits of inverse systems. Recall that an *inverse system* $(X_\lambda, \phi_{\mu\nu})$ is a family of topological spaces (X_λ) indexed by a directed set Λ, and a family of continuous maps $(\phi_{\mu\nu} : X_\nu \to X_\mu)$ with $\mu \leq \nu \in \Lambda$, such that $\phi_{\lambda\lambda} = \mathrm{id}_{X_\lambda}$ and $\phi_{\lambda\mu}\phi_{\mu\nu} = \phi_{\lambda\nu}$ for all $\lambda, \mu, \nu \in \Lambda$ such that $\lambda \leq \mu \leq \nu$.

An *inverse limit* (X, ϕ_λ) of an inverse system $(X_\lambda, \phi_{\mu\nu})$ of topological Lie rings is a topological Lie ring X together with a compatible family of continuous Lie ring homomorphisms $(\phi_\lambda : X \to X_\lambda)$ with the universal property that, for any topological Lie ring Y with a compatible family of homomorphisms $(\psi_\lambda : Y \to X_\lambda)$, there exists a unique map $\psi : Y \to X$ such that $\phi_\lambda \psi = \psi_\lambda$ for each λ.

The inverse limit, which we shall denote $\varprojlim_{\lambda\in\Lambda} X_\lambda$, exists and is unique. In particular it can be realised as the subring of the Cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ formed by elements (x_λ) such that $\phi_{\mu\nu}\pi_\nu(x_\lambda) = \pi_\mu(x_\lambda)$ where $\pi_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu$ is the canonical projection map.

Theorem 1.2.3. *Let $\mathscr{C}$ be a class of finite Lie rings that is closed under Cartesian products, let L be a pro-$\mathscr{C}$ Lie ring, and let $I_{\mathscr{C}} = \{J \lhd_o L | L/J \in \mathscr{C}\}$. Then L is isomorphic to $\varprojlim_{J \in I_{\mathscr{C}}}(L/J)$. Conversely the inverse limit of any inverse system of finite Lie rings that lie in the class $\mathscr{C}$ is a pro-$\mathscr{C}$ Lie ring.*

Proof. It is clear that $(L/J, \phi_{JK})$ is an inverse system for $J \in I_{\mathscr{C}}$, directed by reverse inclusion, and with $\phi_{JK} : L/K \to L/J$ for $K \subseteq J$ being the natural surjection.

Let $\psi : L \to \prod_{J \in I_{\mathscr{C}}} L/J : x \mapsto (x + J)_{J \in I_{\mathscr{C}}}$. It can easily be seen that $\psi(L) \subseteq \hat{L} = \varprojlim_{J \in I_{\mathscr{C}}}(L/J) \leq \prod_{J \in I_{\mathscr{C}}} L/J$. Let $\theta : L \to \hat{L}$ be the induced map. We claim that theta is the required topological Lie ring isomorphism.

For each $K \in I_{\mathscr{C}}$, let $\pi_K : \prod_{J \in I_{\mathscr{C}}} L/J \to L/K$ be the canonical projection. The open sets of the form ${\pi_K}^{-1}(U)$, for $K \in I_{\mathscr{C}}$ and U an open subset of L/K, form a base for the topology of $\hat{L}$. Since, for any such K and U, $\theta^{-1}({\pi_K}^{-1}(U))$ is clearly open in L, it follows that θ is continuous.

Now, for any $K \in I_{\mathscr{C}}$, it is clear that $\pi_K(\theta(L)) = L/K$, and thus, for any set U open in L/K, ${\pi_K}^{-1}(U) \cap \theta(L)$ is non-empty. Since the sets ${\pi_K}^{-1}(U)$ form a base for the topology of $\hat{L}$, it follows that $\theta(L)$ is dense in $\hat{L}$. However, as L is compact, θ continuous, and $\hat{L}$ Hausdorff, $\theta(L)$ is closed and so $\theta(L) = \hat{L}$. Thus θ is surjective.

Finally $\ker \theta \subseteq \bigcap_{I \in I_{\mathscr{C}}} I = 0$, therefore θ is injective, and thus θ is a topological isomorphism as claimed.

For the converse, let $(L_\lambda, \phi_{\mu\nu})$, indexed by the directed set Λ, be the inverse system of Lie rings of class $\mathscr{C}$. Then $\prod_{\lambda \in \Lambda} L_\lambda$ is compact (by Tychonoff's theorem) and Hausdorff (since each L_λ is Hausdorff). It is also clear, from the definition of the product topology, that Lie rings of the form $\prod_{\lambda\in S} L_\lambda \times \prod_{\lambda\notin S}\{0\}$, for S a subset of Λ, will form a neighbourhood base of 0. Since $\mathscr{C}$ is closed under cartesian products it follows that $\prod_{\lambda\in\Lambda} L_\lambda$ is a pro-$\mathscr{C}$ Lie ring. It only remains to show that $\hat{L} = \varprojlim_{\lambda\in\Lambda} L_\lambda$ is a closed subring of $\prod_{\lambda\in\Lambda} L_\lambda$.

Let $(x_\lambda) \in \prod_{\lambda\in\Lambda} L_\lambda \setminus \hat{L}$ and $\pi_\mu : \prod_{\lambda\in\Lambda} L_\lambda \to L_\mu$ be the canonical projection. Then there exist $\mu, \nu \in \Lambda$, with $\nu > \mu$, such that $\phi_{\mu\nu}\pi_\nu(x_\lambda) \neq \pi_\mu(x_\lambda)$, and we can form an open neighbourhood $\prod_{\lambda\in\Lambda\setminus\{\mu,\nu\}} L_\lambda \times \{x_\mu\} \times \{x_\nu\}$ of (x_λ) that is disjoint from $\hat{L}$. It follows that $\prod_{\lambda\in\Lambda} L_\lambda \setminus \hat{L}$ is open, and hence $\hat{L}$ is closed as required. □

Making use of this equivalence we may deduce a further result regarding generation of subrings:

Lemma 1.2.4. *Let L be a pro-finite Lie ring, let K be a closed subring of L, and let d be a positive integer. If $K + I/I$ can be generated by d elements for every $I \lhd_o L$, then $d(K) \leq d$.*

Proof. Let Y_I be the finite set of all d-tuples of generators of $(K + I)/I$. Then $\{Y_I | I \lhd_o L\}$ is an inverse system of compact spaces, directed in a natural way; thus, by Tychonoff's theorem, $\varprojlim_{I\lhd_o L} Y_I$ is non-empty. Therefore there exists $x_1, \ldots, x_d \in K$ such that $x_1 + I, \ldots, x_d + I$ generate $(K + I)/I$ for all $I \lhd_o L$. Thus

$$K = \bigcap_{I\lhd_o L} (\langle x_1, \ldots, x_d\rangle + I) = \overline{\langle x_1, \ldots, x_d\rangle},$$

as required. □

To develop an analog to pro-p groups, we first introduce finite-p Lie rings. We shall call a Lie ring *finite-p* whenever it is finite, nilpotent, and has characteristic a power of p. A *pro-p Lie ring* L is then a compact Hausdorff topological Lie ring with the property that the open ideals I, with L/I finite-p, form a neighbourhood base of 0. That is, L is pro-$\mathscr{C}$ with $\mathscr{C}$ being the class of finite-p Lie rings. Thus from Theorem 1.2.3 a pro-p Lie ring is an inverse limit of an inverse system of finite-p Lie rings.

Note that a pro-p Lie ring is naturally a Lie $\mathbb{Z}_p$-algebra, and conversely every Lie $\mathbb{Z}_p$-algebra that is finitely generated as a $\mathbb{Z}_p$-module contains an open ideal that is a pro-p Lie ring.

The other class of pro-$\mathscr{C}$ Lie rings that we shall be considering is that of pro-nilpotent Lie rings. A *pro-nilpotent Lie ring* is a compact Hausdorff topological Lie ring with the property that the open ideals I, with L/I finite and nilpotent, form a neighbourhood base of 0. Alternatively a pro-nilpotent Lie ring is an inverse limit of an inverse system of finite nilpotent Lie rings.

Chapter 2

The structure and properties of pro-p Lie rings

In this chapter we will concentrate on the class of pro-finite Lie rings that we call pro-p Lie rings that were introduced in the previous chapter. More precisely, we intend to adapt some of the recently developed theory of pro-p groups to a Lie ring setting. In particular, we consider the $\mathbb{Z}_p$-module structure, Prüfer rank, and subring growth of pro-p Lie rings. We also deduce that a finitely generated pro-p Lie ring L is finitely generated as a $\mathbb{Z}_p$-module if and only if, for sufficiently large n, $W_n = \langle e_{12}, te_{22} \rangle \subseteq \mathfrak{gl}_2(\mathbb{F}_p[t]/\langle t^n \rangle)$ is not a closed section of L. These results are then drawn together as a set of equivalent conditions for pro-p Lie rings. Many of our ideas and techniques are drawn from [4]. We apologise that it is sometimes difficult to give appropriate credit to the individual original researchers.

2.1 Frattini theory for pro-p Lie rings

We begin with a brief discussion of the Frattini subring of a pro-p Lie ring. Define the *Frattini subring* $\Phi(L)$ of a pro-p Lie ring L to be the intersection of all (proper) open maximal subrings of L. It is not hard to see that if $K \triangleleft_c L$ such that $K \subseteq \Phi(L)$

then $\Phi(L/K) = \Phi(L)/K$. It can also be shown that if $X + \Phi(L)$ generates L (topologically), then the subset X alone generates L.

We shall also make frequent use of the fact that, for any pro-p Lie ring L, the underlying additive group $(L, +)$ is an Abelian pro-p group.

Proposition 2.1.1. *If L is any pro-p Lie ring then $\Phi(L) = \overline{pL + [L, L]} \lhd_c L$.*

Proof. Consider first the case when L is finite. Let $\pi : L \to L/(pL + [L, L])$ be the natural map, and M_i for $i = 1, 2, \ldots, n$ be the maximal subrings of L. Then $\pi(\bigcap_{i=1}^n M_i) \subseteq \bigcap_{i=1}^n \pi(M_i)$, and the $\pi(M_i)$ are precisely the maximal subrings of $\pi(L)$. As $L/(pL + [L, L])$ is an Abelian Lie $\mathbb{F}_p$-algebra, the intersection of its maximal subrings is 0. So we have $\pi(\bigcap_{i=1}^n M_i) \subseteq \bigcap_{i=1}^n \pi(M_i) = 0$. Thus,

$$\Phi(L) = \bigcap_{i=1}^{n} M_i \subseteq \ker(\pi) = pL + [L, L].$$

Conversely, let M be some maximal subring of L. First we claim that $pL \subseteq M$. If not, then there must exist some $y \in L$ such that $py \notin M$ but $p^2 y \in M$. Since M is maximal, we have $\langle M, py \rangle = L$. But then

$$py \in pL = p\langle M, py \rangle = p(M + \langle py \rangle + [M, py]) \subseteq M;$$

a contradiction. Thus $pL \subseteq M$ as claimed.

Next we claim that $[L, L] \subseteq M$. Indeed, otherwise the nilpotence of L implies the existence of $x \in [L, L]$ such that $x \notin M$ and yet $[x, L] \subseteq M$. This is impossible, however, for then $\langle M, x\rangle = L$ and so

$$x \in [L, L] = [M, M] + [x, M] \subseteq M.$$

Since M was arbitrary, we have $[L, L]+pL \subseteq \Phi(L)$, which when combined with the reverse inclusion proved above yields $\Phi(L) = pL + [L, L]$.

To deduce the general case, let $I \lhd_o L$. Then L/I is finite-p and so $\Phi(L)+I/I \subseteq \Phi(L/I) = (pL + [L, L]) + I/I$. Since $\Phi(L)$ is closed, this yields

$$\Phi(L) \subseteq \bigcap_{I \lhd_o L} pL + [L, L] + I = \overline{pL + [L, L]}.$$

Conversely, let M be a maximal open subring of L. Then there exists $I \lhd_o L$ such that $I \subseteq M$. By the finite case we have

$$pL/I + [L/I, L/I] = \Phi(L/I) \subseteq M/I,$$

and hence $pL + [L, L] \subseteq M$. □

Lemma 2.1.2. *Let $L = \overline{\langle x_1, x_2, \ldots, x_d\rangle}$ be a finitely generated pro-p Lie ring. Then the following statements hold:*

(i) *If $I \lhd_c L$ then $[I, L] \lhd_c L$.*

(ii) *$\gamma_n(L) \lhd_c L$ for all $n \geq 1$.*

Proof. Let $I \lhd_c L$, set $Y = \{\sum_{i=1}^{d}[y_i, x_i] | y_i \in I\} \subseteq [I, L]$, and consider the map

$$\psi : I \times \cdots \times I \to L : (y_1, \ldots, y_d) \mapsto \sum_{i=1}^{d}[y_i, x_i].$$

Clearly ψ is a continuous map from a compact space to a Hausdorff space; consequently, its image Y is closed in L.

Now let $K \lhd_o L$ and note that $L/K = \langle x_1 + K, \ldots, x_d + K \rangle$. Using the Jacobi identity, and the fact that $I \lhd L$, it follows that $[I, L] + K/K = (Y + K)/K$. Therefore we have

$$[I, L] \subseteq \bigcap_{K \lhd_o L} Y + K = \overline{Y} = Y,$$

and hence $[I, L] = Y$ is closed. Part (ii) follows easily from (i). □

Proposition 2.1.3. *Let L be a pro-p Lie ring. Then the following statements hold:*

(i) *L is finitely generated if and only if $\Phi(L)$ is open in L. Furthermore, in this case, $p^{d(L)} = |L/\Phi(L)|$.*

(ii) *If L is finitely generated then $\Phi(L) = pL + [L, L]$.*

Proof. Suppose $L = \overline{\langle x_1, \ldots, x_d \rangle}$ with $d = d(L)$. Then, by Proposition 2.1.1,

$$|L/\Phi(L)| = |L/\overline{pL + [L, L]}| = |\mathbb{F}_p x_1 + \cdots + \mathbb{F}_p x_d| = p^d$$

is finite, and so the closed subring $\Phi(L)$ is open.

Conversely, suppose $\Phi(L) \lhd_o L$ and let X be a finite (additive) transversal of L over $\Phi(L)$. Then $X + \Phi(L)$ generates L, hence X generates L, and so L is finitely generated.

Finally, since $[L, L]$ is closed when L is finitely generated by Lemma 2.1.2, $pL + [L, L]$ is closed, too. □

2.2 The p-adic module structure of pro-p Lie rings

A key notion in the analysis of the p-power structure of pro-p groups is the property of being powerful. Roughly speaking, a pro-p group is said to be powerful whenever it is Abelian modulo its p-powers. Lubotzky and Mann ([9, 10]) made effective use of this concept when they extended the work of Lazard to show that a finitely generated pro-p group is p-adic analytic precisely when it contains an open powerful subgroup. Presently, we shall introduce the analogous notion of a powerful pro-p Lie ring, and consider its impact on the structure of pro-p Lie rings.

We shall call a pro-p Lie ring L *powerful* if $[L, L] \subseteq pL$. We shall say that $I \lhd_c L$ is *powerfully embedded* in L if $[I, L] \subseteq pI$. Observe that, in contrast to powerful groups, no special handling of the prime $p = 2$ is required. Also notice that for powerful L we have $\Phi(L) = \overline{pL + [L, L]} = \overline{pL} = pL$. However, the Frattini subgroup of the underlying additive pro-p group $(L, +)$ is $\Phi(L, +) = pL$ (see [4]); consequently, $\log_p |L/pL| = d(L, +) = d(L)$ and so L is finitely generated as a pro-p Lie ring precisely when $(L, +)$ is finitely generated as a pro-p group.

To group theorists it may be surprising to note that, for every pro-p Lie ring L, the closed ideal pL is *always* powerful: $[pL, pL] \subseteq p(pL)$. More generally, given any $I \lhd_c L$ such that $I \subseteq pL$, we have I powerfully embedded in pL as $[I, pL] = p[I, L] \subseteq pI$.

Next observe that the additive maps

$$p^n L/p^{n+1}L \to p^{n+1}L/p^{n+2}L : x + p^{n+1}L \mapsto px + p^{n+2}L$$

are surjective for all $n \geq 0$. Since $p^n L$ is powerful for $n \geq 1$, we have $p^{d(p^n L)} = |p^n L/\Phi(p^n L)| = |p^n L/p^{n+1} L| = p^{d(p^n L,+)}$, for each $n \geq 1$, and consequently

$$d(L,+) \geq d(pL) = d(pL,+) \geq d(p^2 L) = d(p^2 L,+) \geq \cdots .$$

Proposition 2.2.1. *For any pro-p Lie ring L, the following conditions are equivalent:*

(i) *L is finitely generated as an additive pro-p group.*

(ii) *The ideal pL is open in L.*

(iii) *The ideals $\{p^n L\}_{n \geq 1}$ form a neighbourhood base of 0 consisting of powerful open ideals that are finitely generated as additive pro-p groups.*

(iv) *L contains an open powerful ideal that is finitely generated as a pro-p Lie ring.*

Proof. The equivalence of the first three conditions follows immediately from the preceding discussion. Certainly these imply the fourth. For the converse, suppose that L has an open powerful ideal I that is finitely generated as a pro-p Lie ring. Then, by Proposition 2.1.3, we have

$$pI = pI + [I, I] = \Phi(I) \lhd_o I \lhd_o L.$$

Since $pI \subseteq pL$ and pI is open in L, we have pL open in L. □

There is a natural $\mathbb{Z}_p$-action on a pro-p Lie ring L: if $\lambda \in \mathbb{Z}_p$ and $x \in L$ then $\lambda x = \lim_{n \to \infty} a_n x$ is well defined, where (a_n) is a sequence of integers with

$\lim_{n\to\infty} a_n = \lambda$. Notice that $\mathbb{Z}_p x = \overline{\langle x\rangle}$. It follows that the rank of L as a $\mathbb{Z}_p$-module is precisely $d(L,+)$ (see Proposition 3.7 of [4]), the minimal number of generators of the Abelian pro-p group $(L,+)$.

Lemma 2.2.2. *Let L be a powerful pro-p Lie ring. Then the rank of L as a $\mathbb{Z}_p$-module is $d(L,+) = d(L)$. More precisely, if $L = \overline{\langle x_1, x_2, \ldots, x_d\rangle}$ then $L = \mathbb{Z}_p x_1 + \cdots + \mathbb{Z}_p x_d$.*

Proof. The facts that $d(L,+) = d(L)$ and that $d(L,+)$ coincides with the rank of L as a $\mathbb{Z}_p$-module have already been explained.

For the more specific assertion, we first consider the case when L is finite. Since $[L,L] \subseteq pL$, it follows that $p^t[L,L] \subseteq p^{t+1}L$ and hence

$$p^t L = p^t\langle x_1\rangle + \cdots + p^t\langle x_d\rangle + p^{t+1}L \subseteq \langle x_1\rangle + \cdots + \langle x_d\rangle + p^{t+1}L.$$

This yields $L = \langle x_1\rangle + \cdots + \langle x_d\rangle + p^t L$ for each $t \geq 0$, as required.

For the general case, consider the closed subring $K = \mathbb{Z}_p\, x_1 + \cdots + \mathbb{Z}_p\, x_d$ in L. Now, for any $I \lhd_o L$, the finite case implies $L/I = K + I/I$; thus, $L = \bigcap_{I \lhd_o L} K + I = \overline{K} = K$, as required. □

A finitely generated powerful pro-p Lie ring will called be *uniform* if it satisfies any of the equivalent conditions in the following lemma.

Lemma 2.2.3. *Let L be a finitely generated powerful pro-p Lie ring. The following conditions are equivalent:*

(i) *L is a finitely generated free $\mathbb{Z}_p$-module;*

(ii) $|L : pL| = |p^k L : p^{k+1} L|$, *for all* $k \geq 1$;

(iii) *The pro-p group* $(L, +)$ *is torsion-free.*

Proof. First note that since L is finitely generated and powerful, it follows from Proposition 2.2.1 that the ideals of the form $p^n L$ form a neighbourhood base of 0. Now, in order to show (ii) implies (iii), suppose $(L, +)$ is not torsion-free. It follows that L contains an element x of order p, since any element co-prime to p would have to lie in the same component as 0. From above, there exists an integer k such that $x \in p^k L \setminus p^{k+1} L$; however, then $|p^k L / p^{k+1} L| > |p^{k+1} L / p^{k+2} L|$, as required.

To see why (iii) implies (ii), suppose that, for some k, $|p^k L : p^{k+1} L| > |p^{k+1} L : p^{k+2} L|$. Then there exists some $x \in p^k L \setminus p^{k+1} L$ such that $px = p^{k+2} y$ for some $y \in L$, so that $x - p^{k+1} y$ has order exactly p.

That condition (i) implies (iii) is clear, so it remains only to show that (iii) implies (i). First we employ Lemma 2.2.2 in order to write $L = \mathbb{Z}_p x_1 + \cdots + \mathbb{Z}_p x_d$ for some $x_1, \ldots, x_d \in L$ with $d = d(L, +) < \infty$. Recall also that $d(L, +)$ coincides with the rank of L when viewed as a $\mathbb{Z}_p$-module. It suffices to verify that condition (iii) implies that the generators $x_1, \ldots, x_d$ are free over $\mathbb{Z}_p$. Indeed, suppose $\sum_{i=1}^{d} \lambda_i x_i = 0$, for some $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{Z}_p$, not all 0. This leads easily to the impossibility that either $(L, +)$ has an element of order p or one of the generators is redundant. □

Proposition 2.2.4. *For any pro-p Lie ring* L *the following are all equivalent:*

(i) *The ideal* pL *is open in* L.

(ii) *The ideal* $p^k L$ *is open and uniform for all sufficiently large* k.

(iii) *L contains an open uniform ideal that is finitely generated and free as a $\mathbb{Z}_p$-module.*

(iv) *L is finitely generated as a $\mathbb{Z}_p$-module.*

(v) *L is countably generated as a $\mathbb{Z}_p$-module.*

Proof. First we demonstrate the equivalence of (i) and (ii). Clearly if some $p^k L$ is open then pL is open. Conversely, suppose pL is open in L. Then $|L/pL| = p^{d_0}$ for some integer d_0. Set $|p^n L/p^{n+1}L| = p^{d_n}$ for each $n \geq 1$. Then, as previously remarked, $p^n L$ is powerful for each $n \geq 1$ and $d_0 \geq d_1 \geq \cdots$. Hence there must exist some m such that $d_k = d_m$ for all $k \geq m$; in other words, $p^k L$ is uniform for all $k \geq m$. Since $p^k L$ is closed and of finite index in L, it is open. The equivalence of (iii) to (i) and (ii) now follows from Lemma 2.2.3.

That (iii) implies (iv) is clear. Conversely, suppose that $L = \mathbb{Z}_p x_1 + \cdots + \mathbb{Z}_p x_m$. Then $L/pL = \mathbb{F}_p x_1 + \cdots + \mathbb{F}_p x_m$ and so $|L/pL| \leq p^m$ and hence pL is open.

Finally, we demonstrate that if L is countably generated as a $\mathbb{Z}_p$-module then it is finitely generated. Suppose L is generated by a countably infinite set $X = \{x_1, x_2, \ldots\}$. Let $X_s = \{x_1, \ldots, x_s\}$ and let $M(X_s)$ denote the $\mathbb{Z}_p$-submodule of L generated by X_s. Then $L = \bigcup_{s=1}^{\infty} M(X_s)$, and so by the Baire Category Theorem there exists s_0, $w \in L$ and $I \lhd_o L$ such that $w + I \subseteq M(X_s)$ for all $s \geq s_0$. Let $t_1, \ldots, t_m$ be a transversal of I in L. Then $L = M(X_s) + \mathbb{Z}_p w + \mathbb{Z}_p t_1 + \cdots + \mathbb{Z}_p t_m$ and so L is finitely generated as a $\mathbb{Z}_p$-module. □

A p-adic analytic group has the structure of a p-adic Lie group and so, via the Lie algebra correspondence and the Ado-Iwasawa theorem, is a linear group; that is, a p-adic analytic pro-p group is necessarily isomorphic to a closed subgroup of $GL_n(\mathbb{Z}_p)$ for some n. The following result of Weigel ([27]), which extends the Ado-Iwasawa theorem, combined with the results of this section regarding the $\mathbb{Z}_p$-module structure of pro-p Lie rings, allows us to establish an analogous Lie theoretic result.

Theorem 2.2.5 (Weigel ([27])). *Let R be a principal ideal domain such that, for all non-trivial ideals $P \lhd R$, R/P has positive characteristic, and let L be a Lie R-algebra that is finitely generated as an R-module. Then there exists a finitely generated R-module V and a faithful R-linear representation $\phi : L \to \mathfrak{gl}(V)$.*

Theorem 2.2.6. *A pro-p Lie ring L is finitely generated as a $\mathbb{Z}_p$-module if and only if L is isomorphic to a closed subring of $\mathfrak{gl}(V)$ for some finitely generated $\mathbb{Z}_p$-module V.*

Proof. Let V be finitely generated as a $\mathbb{Z}_p$-module, and let $V = U \oplus W$ such that U is free as a $\mathbb{Z}_p$-module and W, the torsion part of V, is finite. Since, for any $\phi \in \mathfrak{gl}(V)$ and $w \in W$, $\phi(w) \in W$, there exists a well defined map $\psi : \mathfrak{gl}(V) \to \mathfrak{gl}(V/W)$ where $\psi \circ \phi(v + W) = \phi(v) + W$. Furthermore, any element of $\ker(\psi)$ is completely determined by its action on a finite generating set of U and the elements of W, and therefore $\ker(\psi)$ is finitely generated as a $\mathbb{Z}_p$-module. Now since both $\mathfrak{gl}(V/W)$ and $\ker(\psi)$ are finitely generated as $\mathbb{Z}_p$-modules it follows that $\mathfrak{gl}(V)$ is as well. Therefore if L embeds into $\mathfrak{gl}(V)$ then L itself must be finitely generated as a $\mathbb{Z}_p$-module.

For the converse, suppose that L is finitely generated as a $\mathbb{Z}_p$-module and note that, since $\mathbb{Z}_p$ is a principal ideal domain satisfying the required condition of Theorem 2.2.5, there exists a finitely generated $\mathbb{Z}_p$-module V, and an embedding $\phi : L \hookrightarrow \mathfrak{gl}(V)$. It therefore remains only to show that ϕ is continuous.

Clearly $(L, +)$ is a finitely generated pro-p group, and because $\mathfrak{gl}(V)$ inherits a $\mathbb{Z}_p$-action from V it follows that, additively, $\mathfrak{gl}(V)$ has the structure of an Abelian pro-p group and hence is pro-finite. Now, by Corollary 1.2.1 of [4], we have all homomorphisms from finitely generated pro-p groups to profinite groups being necessarily continuous, and hence ϕ is continuous as required. □

We close this section with an example constructing a pro-p Lie ring from a Lie $\mathbb{Z}_p$-algebra, and using results of this section to infer results about the structure of the original Lie $\mathbb{Z}_p$-algebra.

Example. Consider any Lie $\mathbb{Z}_p$-algebra L such that L is isomorphic to $\mathbb{Z}_p^d$ as a $\mathbb{Z}_p$-module. Then pL is a d-generated powerful pro-p Lie ring that is open in L. To establish this, we must first verify that pL is a topological Lie ring. Here pL inherits its topology from the pro-p group $(L, +)$. Next we claim that $L \times L \to L : (x, y) \mapsto [x, y]$ is continuous. Let H be an open subgroup of $(L, +)$. Then $(L, +)/H$ is a finite p-group, so there exists t such that $p^t L \subseteq H$. Since $[p^t L, L] \subseteq p^t L \subseteq H$, it follows that the inverse image of H contains $p^t L \times L$ and is therefore open in $L \times L$ as claimed. Now, because $\{p^t L\}_{t \geq 1}$ forms a neighbourhood base of 0 in pL, it is clear that pL is a pro-p Lie ring. Finally, by observations at the beginning of the section, pL is powerful. An immediate corollary is that L itself contains an open uniform pro-p Lie ring of the form $p^k L$ for some positive integer k.

Similarly, any Lie $\mathbb{Z}_p$-algebra L that is finitely generated as a $\mathbb{Z}_p$-module contains an open uniform pro-p Lie ring: we may write $L = U \oplus W$ with W the torsion part of L, and then, for some $k \geq 1$, we have $p^k L = p^k U$ an open powerful torsion free ideal of L.

2.3 Prüfer rank

Lubotzky and Mann demonstrated that whether or not a pro-p group has finite Prüfer rank has a fundamental impact on its underlying structure. For example, they showed in [10] that having finite Prüfer rank is precisely equivalent to being p-adic analytic. Following their work and closely related results in [4], we seek to extend the characterisations of pro-p Lie rings begun in the previous section.

Lemma 2.3.1. *The following possible definitions of the* Prüfer rank $\mathrm{rk}(L)$ *of a pro-p Lie ring L coincide:*

(i) $\mathrm{rk}_1(L) = \sup\{d(K) | K \leq_c L\}$.

(ii) $\mathrm{rk}_2(L) = \sup\{d(K) | K \leq_c L \text{ and } d(K) < \infty\}$.

(iii) $\mathrm{rk}_3(L) = \sup\{d(K) | K \leq_o L\}$.

(iv) $\mathrm{rk}_4(L) = \sup\{\mathrm{rk}(L/I) | I \lhd_o L\}$.

Proof. Let $I \lhd_o L$ and $K/I \leq L/I$. Then clearly

$$d(K/I) \leq d(K) \leq \sup\{d(K) | K \leq_o L\}$$

and hence $\mathrm{rk}_4(L) \leq \mathrm{rk}_3(L)$. We also have $K = I + X$, for some finite subset X of L, and so

$$d(K/I) \leq d(\overline{\langle X \rangle} + I/I) \leq d(\overline{\langle X \rangle}) \leq \sup\{d(K) | K \leq_c L \text{ and } d(K) < \infty\},$$

giving $\mathrm{rk}_4(L) \leq \mathrm{rk}_2(L)$. It is clear that $\mathrm{rk}_3(L) \leq \mathrm{rk}_1(L)$ and $\mathrm{rk}_2(L) \leq \mathrm{rk}_1(L)$, so it remains only to show that $\mathrm{rk}_1(L) \leq \mathrm{rk}_4(L)$. Indeed, if $K \leq_c L$ then

$$d(K) \leq \sup\{d(K+I/I) | I \lhd_o L\} \leq \sup\{\mathrm{rk}(L/I) | I \lhd_o L\}$$

by Lemma 1.2.4. □

Proposition 2.3.2. *If L is a pro-p Lie ring with finite Prüfer rank r then $\gamma_{2r}(L) \subseteq pL$, and hence pL is open in L.*

Proof. Let $L_p = L/pL$ and consider that $\mathrm{rk}(L_p) \leq \mathrm{rk}(L) = r$. Now since the closed section $\gamma_r(L_p)/\gamma_{2r}(L_p)$ is an Abelian Lie $\mathbb{F}_p$-algebra, we have

$$\sum_{i=0}^{r-1} \dim_{\mathbb{F}_p}(\gamma_{r+i}(L_p)/\gamma_{r+i+1}(L_p)) = \dim_{\mathbb{F}_p}(\gamma_r(L_p)/\gamma_{2r}(L_p)) \leq r.$$

Thus, we can assume that each $\dim_{\mathbb{F}_p}(\gamma_{r+i}(L_p)/\gamma_{r+i+1}(L_p)) = 1$, for otherwise we have $\gamma_{r+i}(L_p) = \gamma_{r+i+1}(L_p) = 0$ for some $i < r$ and hence $\gamma_{2r}(L) \subseteq pL$. In particular, we have $\gamma_r(L_p) = \mathbb{F}_p x + \gamma_{r+1}(L_p)$ for some $x \in \gamma_r(L_p)$. Consequently, the section $\gamma_r(L_p)/\gamma_{2r+1}(L_p)$ is an Abelian Lie $\mathbb{F}_p$-algebra, and so $\dim_{\mathbb{F}_p}(\gamma_r(L_p)/\gamma_{2r+1}(L_p)) \leq r$. However, by assumption, $\dim_{\mathbb{F}_p}(\gamma_{r+i}(L_p)/\gamma_{r+i+1}(L_p)) = 1$ for each $0 \leq i < r$, and so we are forced to have $\gamma_{2r}(L_p) = \gamma_{2r+1}(L_p) = 0$; in other words, $\gamma_{2r}(L) \subseteq pL$. Finally, since L is finitely generated, $L/pL = L/(pL + \gamma_{2r}(L))$ is finite, and so the closed ideal pL is open. □

Theorem 2.3.3. *Let L be a finitely generated powerful pro-p Lie ring. Then $\mathrm{rk}(L) = d(L)$. Consequently, the Prüfer rank of L coincides precisely with rank of L as a $\mathbb{Z}_p$-module, which further coincides with $d(L,+)$ and $\mathrm{rk}(L,+)$, the Prüfer rank of the pro-p group $(L,+)$.*

Proof. From Lemma 2.2.2, we know $d(L,+) = d(L)$ and so $(L,+)$ is a finitely generated Abelian pro-p group. It follows that $\mathrm{rk}(L,+) = d(L,+)$ (see Theorem 3.8 in [4]). Thus, we need only show $\mathrm{rk}(L) \leq d(L)$ for then

$$\mathrm{rk}(L) = d(L) = d(L,+) = \mathrm{rk}(L,+).$$

According to Lemma 2.3.1, it suffices to prove $\mathrm{rk}(L/I) \leq d(L/I) \leq d(L)$, for each $I \lhd_o L$. Thus, we may assume L is finite-p.

We define the weight of each $x \in L$ by

$$\nu(x) = \begin{cases} \max\{j | x \in p^j L\}, & x \neq 0 \\ \infty, & x = 0 \end{cases}.$$

For a finite subset $X \subseteq L$, we define

$$\nu(X) = \sum_{x \in X} \nu(x).$$

Given any $K \leq L$, we need to show that $d(K) \leq d(L)$. So, let us assume, to the contrary, that $d(L) < d(K) = k$, and among all minimal generating sets X of K choose one such that $\nu(X)$ is maximal. Let $X = \{x_1, \dots, x_k\}$, where $\nu(x_1) \leq \nu(x_2) \leq \cdots \leq \nu(x_k)$. Next pick a set $Y = \{y_1, \dots, y_k\}$ of weight 0 such that $p^{\nu(x_i)} y_i = x_i$ for each i. Recall $\dim_{\mathbb{F}_p} L/\Phi(L) = d(L) < k$, so that Y is a linearly dependent set modulo $\Phi(L)$. Let m be minimal such that y_m lies in the $\mathbb{F}_p$-span of $\{y_1, \dots, y_{m-1}\}$ modulo $\Phi(L)$ and write $y_m = w + \sum_{i=1}^{m-1} \alpha_i y_i$ where

$w \in \Phi(L)$. Then

$$x_m = p^{\nu(x_m)} w + \sum_{i=1}^{m-1} p^{\nu(x_m)} \alpha_i y_i.$$

However, $\nu(x_m) \geq \nu(x_i)$ for $m \geq i$ and so

$$x_m = p^{\nu(x_m)} w + \sum_{i=1}^{m-1} p^{\nu(x_m) - \nu(x_i)} \alpha_i x_i.$$

Thus $x_m = x + z$ for some $x \in \langle x_1, \ldots, x_{m-1} \rangle$ and $z \in p^{\nu(x_m)} \Phi(L) = p^{\nu(x_m)+1} L$ as L is powerful. Hence $\langle x_1, \ldots, x_{m-1}, z, x_{m+1} \ldots, x_k \rangle = K$. However this generating set has strictly greater weight than X, contradicting our choice of X. $\square$

Theorem 2.3.4. *For any pro-p Lie ring L, the following conditions are equivalent.*

(i) *L has finite Prüfer rank.*

(ii) *pL is open in L.*

(iii) *The pro-p group $(L, +)$ has finite Prüfer rank.*

Proof. From Proposition 2.3.2, we have (i) implies (ii). If pL is open then $p^{d(L,+)} = |L/pL|$ is finite. Since $(L, +)$ is a finitely generated Abelian pro-p group, it follows (as mentioned above) that $\mathrm{rk}(L, +) = d(L, +)$, and so (ii) implies (iii). The fact that (iii) implies (i) is clear. $\square$

Since L has finite Prüfer rank precisely when pL is open, Proposition 2.2.4 informs us that such an L contains an open uniform ideal I. By Theorem 2.3.3, the Prüfer rank r of I coincides with the rank of I as a $\mathbb{Z}_p$-module. Since L/I is a finite $\mathbb{Z}_p$-module, it follows that the rank of L as a $\mathbb{Z}_p$-module is also r. Thus, the

Prüfer rank of *every* uniform open ideal in L must be r. We define the *uniform dimension* of L to be this uniquely determined integer r.

We close this section with a variety of explicit bounds.

Proposition 2.3.5. *Let L be a pro-p Lie ring with $r = \mathrm{rk}(L)$ and $s = d(L,+)$. Then the following relationships hold.*

(i) $|L/p^k L| \leq p^{(k-1)r+s}$ *for all* $k \geq 1$.

(ii) $r \leq s \leq r(\lceil \log_2 r \rceil + 1)$.

(iii) *If L has finite characteristic p^t then L is finite with* $|L| \leq p^{r(\lceil \log_2 r \rceil + t)}$ *and nilpotence class* $c \leq (2r-1)t$.

Proof. To see why (i) holds, recall from the beginning of Section 2.2 that $|p^n L/p^{n+1}L| = p^{d(p^n L)} \leq p^r$, for all $n \geq 1$, and thus

$$|L/p^k L| = \prod_{n=0}^{k-1} |p^n L/p^{n+1} L| \leq p^s (p^r)^{k-1},$$

as required.

To prove (ii), consider $L_p = L/pL$. Clearly $\mathrm{rk}(L_p) \leq \mathrm{rk}(L) = r$, and so, by Proposition 2.3.2, we have $\gamma_{2r}(L_p) \subseteq pL_p = 0$. Thus, L_p is a Lie $\mathbb{F}_p$-algebra with nilpotence class at most $2r-1$. Since $\gamma_{2^n}(L_p)/\gamma_{2^{n+1}}(L_p)$ is an Abelian Lie $\mathbb{F}_p$-algebra, it follows that $\dim_{\mathbb{F}_p}(\gamma_{2^n}(L_p)/\gamma_{2^{n+1}}(L_p)) \leq r$. We therefore have

$$s = \dim_{\mathbb{F}_p}(L_p) = \sum_{n=0}^{\lceil \log_2((2r-1)+1) \rceil} \dim_{\mathbb{F}_p}(\gamma_{2^n}(L_p)/\gamma_{2^{n+1}}(L_p)) \leq r(\lceil \log_2 r \rceil + 1).$$

Clearly we also have

$$r = \mathrm{rk}(L) \leq \mathrm{rk}(L,+) = d(L,+) = s.$$

Now, to prove (iii), suppose L has finite characteristic p^t. Since L has finite Prüfer rank r, by Proposition 2.3.2, we have $\gamma_{2r}(L) \subseteq pL$. It follows by induction that $\gamma_{2rt-(t-1)}(L) \subseteq p^t L = 0$ and hence $c \leq (2r-1)t$. Finally, using (i) and the bound on s from (ii), we see $|L| = |L/p^t L| \leq p^{r(\lceil \log_2 r \rceil + t)}$, as required. □

Example. Consider $L = \langle e_{12}, p^s e_{22} \rangle \subseteq \mathfrak{gl}_2(\mathbb{Z}_p)$ for some fixed positive integer s. Then L is a uniform pro-p Lie ring of Prüfer rank 2 that is neither finite nor nilpotent. Note that the pro-p group $(L,*)$ corresponding to L under the Baker-Campbell-Hausdorff formula has the presentation $\langle x, y | (x,y)x^{-p^s} \rangle$, where $(x,y) = x^{-1}y^{-1}xy$ is the group commutator.

2.4 Subring growth

Given any pro-p Lie ring L, we shall denote by $\sigma_k(L)$ the number of closed subrings of L with additive index less than or equal to p^k. We shall say that L has *polynomial subring growth* if the sequence $\sigma_k(L)$ grows at most polynomially in p^k.

During the past twenty years or so, subgroup growth of both groups and profinite groups has been one of the most active areas in group theory. See [12] for a detailed overview of this topic. Analogous growth conditions have been studied in other algebraic categories: see [15] and [18] for the category of restricted Lie algebras and [20, 19] for commutative rings.

One seminal result, proved by Lubotzky and Mann in [11], is that a pro-p group has polynomial subgroup growth if and only if it has finite Prüfer rank. The main purpose of this present section is to prove the analogue of this result for pro-p Lie rings.

Let L be a finitely generated pro-p Lie ring. We shall make use of a descending central series of ideals L defined by

$$D_n(L) = \sum_{ip^j \geq n} p^j \gamma_i(L),$$

for each $n \geq 1$. It follows from Lemma 2.1.2 that each $D_n(L)$ is closed and thus open since clearly $L/D_n(L)$ is finite. Observe further that this series satisfies the properties $[D_m(L), D_n(L)] \subseteq D_{m+n}(L)$ and $pD_m(L) \subseteq D_{pm}(L)$, for each $m, n \geq 1$.

Lemma 2.4.1. *Let L be any finitely generated pro-p Lie ring. If there exists some $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that*

$$d(D_{2^n}(L)) < (1 - \epsilon) \log_p |L : D_{2^n}(L)|$$

for all $n \geq n_0$, then pL is open in L.

Proof. Let $e_n = d(D_{2^n}(L))$ and $i_n = \log_p |L : D_{2^n}(L)|$. Since $D_{2^{n+1}}(L)$ is closed, Proposition 2.1.1 yields

$$\Phi(D_{2^n}(L)) = \overline{pD_{2^n}(L) + D_{2^n}(L)'} \subseteq D_{2^{n+1}}(L).$$

Hence,

$$p^{e_n} = |D_{2^n}(L)/\Phi(D_{2^n}(L))| \geq |D_{2^n}(L)/D_{2^{n+1}}(L)| = p^{i_{n+1}-i_n}.$$

If we suppose that $e_n < (1-\epsilon)i_n$, for all $n \geq n_0$, then we must have $i_{n+1} < (2-\epsilon)i_n$ for all $n \geq n_0$. By induction, it follows that $i_{n_0+m} < (2-\epsilon)^m i_{n_0} < 2^{n_0+m} - 1$. If $D_{k+1}(L)$ is properly contained in $D_k(L)$ for all $k < 2^{n_0+m}$ then $i_{n_0+m} \geq 2^{n_0+m} - 1$; a contradiction. Hence, there exists some $k < 2^{n_0+m}$ such that $D_k(L) = D_{k+1}(L)$, and so $\gamma_k(L) \subseteq pL$. This proves pL is open in L. □

Proposition 2.4.2. *Let L be any pro-p Lie ring. If $\sigma_k(L) \leq p^{ck^2}$ for all sufficiently large k, where $c < 1/8$, then pL is open in L. Consequently, if L has polynomial subring growth then L has finite Prüfer rank.*

Proof. First note that, since $\sigma_1(L)$ is finite, the number of closed maximal subrings in L is finite by Proposition 2.1.1; therefore, by Proposition 2.1.3, L is finitely generated since $\Phi(L)$ is open in L.

Suppose, to the contrary, that pL is not open. Fix $\epsilon > 0$. By Lemma 2.4.1, it follows that $e_n = d(D_{2^n}(L)) \geq (1-\epsilon)i_n$, for infinitely many values of n. Let us select such an n and subsequently choose an integer r such that

$$3r \leq (1-\epsilon)i_n \leq 3(r+1).$$

The e_n-dimensional $\mathbb{F}_p$- space $D_{2^n}(L)/\Phi(D_{2^n}(L)$ will then have at least $p^{r(e_n-r)} \geq p^{2r^2}$ subspaces of codimension r. Each such subspace corresponds to a subring of

L of index p^{i_n+r}. Let $k = i_n + r$. It follows that $\sigma_k(L) \geq p^{2r^2}$, where

$$2r^2 \geq 2\left(\frac{(1-\epsilon)k-3}{4-\epsilon}\right)^2 > ck^2$$

for ϵ sufficiently small and k sufficiently large. Since $k \geq i_n \geq n$ and n was arbitrary, we find that, for all sufficiently large k, $\sigma_k(L) > p^{ck^2}$, which is our desired contradiction. The rest is a consequence of Theorem 2.3.4. □

Conversely, if pL is open in L then L has polynomial subring growth:

Proposition 2.4.3. *Let L be any pro-p Lie ring. If pL is open in L then $\sigma_k(L) \leq \sigma_k(L,+) \leq p^{k(s+1)}$, for each $k \geq 1$, where $s = d(L,+) < \infty$.*

Proof. Clearly we always have $\sigma_k(L) \leq \sigma_k(L,+)$. Next notice that $\Phi(L,+) = pL$, so that pL is open in L if and only if $|L/pL| = p^s < \infty$. Since $(L,+)$ is a finitely generated Abelian pro-p group, it now follows by Theorems 3.8 and 3.19 of [4] that $\sigma_k(L) \leq \sigma_k(L,+) \leq p^{k(s+1)}$. □

2.5 Engelian sections

One way of formulating Zelmanov's solution of the restricted Burnside problem for groups ([31]) is the statement: every finitely generated pro-p group of finite exponent is finite. Shalev gave an interesting extension of this result in [23] when he proved that every finitely generated pro-p group of finite Prüfer rank cannot involve all wreath products of the form $C_p \wr C_{p^n}$ $(n \geq 1)$ as closed sections.

Our present goal is to provide an analogous result for pro-p Lie rings. Our proof requires another deep result of Zelmanov which we formulate as follows. Given a positive integer n, a Lie algebra L is said to be n-*Engelian* if $(\mathrm{ad}\ y)^n = 0$ for all $y \in L$. It follows from a classical result of Engel that every finite-dimensional n-Engelian Lie algebra is nilpotent. Zelmanov proved that this assertion remains true under the assumption that L is merely finitely generated (see [31]). By considering the relatively free d-generated n-Engelian Lie $\mathbb{F}$-algebra, it follows that the nilpotence class $c = \mathrm{cl}(L)$ of every d-generated n-Engelian Lie $\mathbb{F}$-algebra L is $\{d, n, \mathbb{F}\}$-bounded; that is to say, there exists a function f such that $c \leq f(d, n, \mathbb{F})$.

Our simple first step seems interesting in its own right:

Theorem 2.5.1. *Let d and n be positive integers and set $m = f(d, 2n-1, \mathbb{F})$. Suppose that L is a nilpotent d-generated Lie $\mathbb{F}$-algebra with the property that $(\mathrm{ad}\ y)^n$ acts trivially on the Abelian section $\gamma_{m+1}(L)/\gamma_{m+1}(L)'$ for all $y \in L$. Then L has nilpotence class at most $2m$.*

Proof. Let $V = \gamma_{m+1}(L)/\gamma_{m+1}(L)'$ and define $L \to \mathrm{End}(V) : y \mapsto \phi_y$, where $\phi_y(v) = [v, y]$. Let $C_L(V)$ denote the centraliser of V in L. Then $L_V = L/C_L(V)$ embeds in $\mathrm{End}(V)$. Now, for any $\phi \in \mathrm{End}(V)$ such that $\phi^n = 0$, it is well-known that $(\mathrm{ad}\ \phi)^{2n-1} = 0$, so $(\mathrm{ad}\ \phi_y)^{2n-1} = 0$. Consequently, L_V is $(2n-1)$-Engelian. Therefore, $\mathrm{cl}(L_V) \leq f(d, 2n-1, \mathbb{F}) = m$.

Now, because the action of L on V is identical to that of L_V, we must have

$$\gamma_{2m+1}(L) = [\gamma_{m+1}(L),_m L] \subseteq \gamma_{m+1}(L)' \subseteq \gamma_{2m+2}(L);$$

but L is nilpotent and thus $\mathrm{cl}(L) \leq 2m$. □

We denote by $W_n = W_n(\mathbb{F})$ the Lie $\mathbb{F}$-algebra $\langle x, y\rangle$ given by the relations $[x,_n y] = 0$ and $[[x,_i y], [x,_j y]] = 0$ for all $0 \leq i, j < n$. This metabelian n-Engelian

Lie algebra can be realised in $\mathfrak{gl}_2(\mathbb{F}[t]/\langle t^n\rangle)$ by taking $x = e_{12}$ and $y = te_{22}$.

Lemma 2.5.2. *Let n be a positive integer and let L be a nilpotent Lie $\mathbb{F}$-algebra. If there exists $y \in L$ and $I \lhd L$ such that $(\mathrm{ad}\ y)^n$ does not act trivially on the Abelian section I/I' then L has a section isomorphic to W_n.*

Proof. By hypothesis, there exists some $x \in I$ such that $[x,_n y] \notin I'$. Since L is nilpotent, we may replace x by some $[x,_i y]$ to assume that $[x,_{n-1} y] \notin I'$ but $[x,_n y] \in I'$. Then $\langle x, y\rangle + I'/I' \cong W_n$. □

Combining the last two results yields the following corollary.

Corollary 2.5.3. *Let n be a positive integer and let L be a nilpotent d-generated Lie $\mathbb{F}$-algebra. If L does not have a section isomorphic to W_n then the nilpotence class of L is $\{d, n, \mathbb{F}\}$-bounded.*

We are ready to apply our results to pro-p Lie rings. We shall say that a pro-p Lie ring L *involves* another pro-p Lie ring K if there exists a closed section of L that is isomorphic to K.

Theorem 2.5.4. *Let L be a finitely generated pro-p Lie ring.*

(i) *Let n be a positive integer and suppose that L does not involve $W_n = W_n(\mathbb{F}_p)$. Then the Prüfer rank of L is finite. In fact, $\mathrm{rk}(L)$ is $\{d(L), n, p\}$-bounded.*

(ii) *Conversely, if $r = \mathrm{rk}(L)$ is finite then L does not involve W_{r+1}.*

Proof. (i) Let K be any open image of L/pL. By hypothesis, K cannot involve W_n and $d(K) \le d(L)$. Thus, by the previous corollary, the nilpotence class of K is $\{d(L), n, p\}$-bounded. Since K was arbitrary, it follows that the nilpotence class of

L/pL is $\{d(L), n, p\}$-bounded, and hence so is $|L/pL|$. The conclusion now follows from results proved in Section 2.3.

(ii) Suppose that L involves W_n. Then $\mathrm{rk}(W_n) \leq r$. However, the ideal in W_n generated by x is Abelian of dimension n over $\mathbb{F}_p$; hence, $n \leq r$. □

2.6 Main Theorem

The primary results of the previous sections are collected into a list of equivalent conditions in the following theorem:

Theorem A. *Let L be a pro-p Lie ring. The following conditions are equivalent:*

(i) *L has finite Prüfer rank.*

(ii) *The pro-p group $(L, +)$ has finite Prüfer rank.*

(iii) *The ideal pL is open in L.*

(iv) *L contains an open powerful ideal that is finitely generated as a pro-p Lie ring.*

(v) *The ideals $\{p^n L\}_{n \geq 1}$ form a neighbourhood base of 0 consisting of powerful open ideals that are each finitely generated as additive pro-p groups.*

(vi) *L is finitely generated and $\gamma_n(L) \subseteq pL$ for some n.*

(vii) *$|L/p^k L|$ grows at most polynomially with k.*

(viii) *L contains an open uniform ideal.*

(ix) *L is finitely generated as a $\mathbb{Z}_p$-module.*

(x) *L is countably generated as a $\mathbb{Z}_p$-module.*

(xi) *Given any $c < 1/8$, $\sigma_k(L) \leq p^{ck^2}$ for all sufficiently large k.*

(xii) *L has polynomial subring growth.*

(xiii) *The pro-p group $(L, +)$ has polynomial subgroup growth.*

(xiv) *L is isomorphic to a closed subring of $\mathfrak{gl}(V)$ for some finitely generated $\mathbb{Z}_p$-module V.*

(xv) *L is finitely generated and does not involve $W_n(\mathbb{F}_p)$ for some $n \geq 1$.*

Corollary 2.6.1. *If L is a pro-p Lie ring with finite Prüfer rank then every subgroup of $(L, +)$ of finite index is open in L. Consequently the topology of such an L is completely determined by its additive structure.*

Proof. Since L has finite Prüfer rank, $(L, +)$ is finitely generated as a pro-p group. Hence, by Theorem 1.17 of [4], every additive subgroup of finite index in $(L, +)$ is open. □

Note that in Chapter 3 we will prove the more general result that every ideal of finite index in a finitely generated pro-nilpotent Lie ring L is necessarily open. Thus, its topology is entirely determined by its algebraic structure.

Chapter 3

The Strong Completeness of pro-finite Lie rings

In Chapter 2 it was observed that for a particular class of pro-finite Lie rings, referred to as pro-p Lie rings with finite rank, the topological structure of the Lie ring was uniquely determined by its algebraic structure. This recalls a result by Serre (see [22]) that the topological structure of a (topologically) finitely generated pro-p group is uniquely determined by its algebraic structure; in other words, every subgroup of finite index is open. Serre's result was expanded upon by Hartley ([6]), and later by Segal, first to all finitely generated pro-soluble groups ([21]), and finally (with Nikolov) to all finitely generated pro-finite groups ([13]). A natural question therefore arises.

> *Is the topology of every finitely generated pro-finite Lie ring uniquely determined by its algebraic structure?*

As a first step, we give a positive solution for the case of all finitely generated pro-nilpotent Lie rings.

A finitely generated pro-finite Lie ring L is said to be *strongly complete* if every ideal of finite index in L is open in L. Our main result is as follows:

Theorem B. *Every finitely generated pro-nilpotent Lie ring is strongly complete. Thus every finite homomorphic image of L is nilpotent and the topology of a finitely generated pro-nilpotent Lie ring is uniquely determined by its algebraic structure.*

3.1 Proof of Theorem B

Our approach necessarily differs from the group case for two reasons. Firstly, subrings of finite index in a finitely generated Lie ring need not be finitely generated themselves. Secondly, it remains an open question as to whether, for Lie rings, subrings of finite index contain ideals of finite index. We deal with these issues by expanding the problem to the category of modules over Lie rings (see [1], for example).

Let L be a Lie ring. An Abelian group $(M, +)$ is an L-*module* if it is equipped with an L-action $M \times L \to M : (a, x) \mapsto [a, x]$ such that

(i) $[a + b, x] = [a, x] + [b, x]$,

(ii) $[a, x + y] = [a, x] + [a, y]$, and

(iii) $[a, [x, y]] = [[a, x], y] - [[a, y], x]$,

for all $a, b \in M$ and $x, y \in L$, where $[x, y]$ also denotes the Lie bracket in L. The L-module M will be referred to as *nilpotent* whenever the L-action is nilpotent. That is to say, there exists a positive integer n such that $[M,_n L] = 0$. To distinguish the manner of generation, we will use $\langle X \rangle$ to denote the Lie subring generated by $X \subseteq L$, while $\langle Y \rangle_L$ will denote the L-submodule generated by $Y \subseteq M$ under addition and L-action, and $\langle Y\rangle_+$ to denote the additive subgroup generated by $Y \subseteq M$.

Now, let L be a topological Lie ring. Then M is a *topological L-module* if M has a topology such that both addition in M and the L-action are continuous. A compact Hausdorff topological L-module M is called *pro-finite* if the open submodules of finite index form a neighbourhood base of 0; such an L-module will be called *pro-nilpotent* if the open submodules $N \leq_o M$, with M/N nilpotent, form a neighbourhood base of 0. We shall say a pro-finite L-module M is (topologically) finitely generated if there exists a finite set $Y \subseteq M$ such that $M = \overline{\langle Y\rangle_L}$.

We will later use the obvious fact that a pro-finite Lie ring L is a pro-finite L-module over itself. The results regarding pro-finite Lie rings discussed in Chapter 1 can be modified for the category of pro-finite L-modules with analogous proofs:

Lemma 3.1.1. *Let L be a pro-finite Lie ring, and let M be a pro-finite L-module. Then:*

(i) *if $N \leq_c M$ then N is a pro-finite L-module;*

(ii) *if M is pro-nilpotent and $N \leq_c M$ then N is a pro-nilpotent L-module;*

(iii) *if Y is a subset of M then*

$$\bigcap_{N \leq_o M} Y + N = \overline{Y}; \ \textit{and,}$$

(iv) *if $K \leq M$ is an (abstract) submodule then*

$$\overline{K} = \bigcap \{N | K \leq N \leq_o M\}.$$

The *Frattini submodule* (i.e. radical), denoted $\Phi_L(M)$, of a pro-finite L-module M is the intersection of all maximal (proper) open submodules of M.

Lemma 3.1.2. *Let L be a pro-finite Lie ring, and let M be a pro-finite L-module. For a subset Y of M, if $\overline{\langle Y \rangle_L} + \Phi_L(M) = M$ then $\overline{\langle Y \rangle_L} = M$.*

Proof. Suppose that $\overline{\langle Y \rangle_L} + \Phi_L(M) = M$. Now, there exists some open submodule K such that $\langle Y \rangle_L \leq K \leq_o M$. Suppose that $K \neq M$; then there exists some maximal proper open submodule N such that $K \subseteq N$. We then have

$$\overline{\langle Y \rangle_L} + \Phi_L(M)/\Phi_L(M) \leq N/\Phi_L(M) < M/\Phi_L(M),$$

which contradicts our supposition, hence $K = M$. However, from Part ((iv)) of Lemma 3.1.1 we then have

$$\overline{\langle Y \rangle_L} = \bigcap \{K | \langle Y \rangle_L \leq K \leq_o M\} = M,$$

as claimed. □

Lemma 3.1.3. *Let L be a pro-finite Lie ring and let M be a pro-nilpotent L-module. Then $\overline{[M, L]} \subseteq \Phi_L(M)$.*

Proof. Since $\Phi_L(M)$ is closed it is sufficient to show that that $[M, L] \subseteq \Phi_L(M)$.

So, suppose that there exists N, a maximal open submodule of M, such that $[M, L] \nsubseteq N$; that is, $[M/N, L] \neq 0$. Then since L acts nilpotently on M/N, there exists $a \in M$ such that $a \notin N$ but $[a, L] \subseteq N$. Furthermore, because $|M/N| = m$, where m is finite, $ma \in N$. Now, by the maximality of N, we have

$$M = \langle N, a \rangle_L = \bigcup_{k=0}^{m-1} (ka + N).$$

Hence $[M, L] \subseteq N$, a contradiction. □

Lemma 3.1.4. *Let L be a pro-finite Lie ring and let M be a finite topological L-module (with the discrete topology). Then the centralizer of M in L, $C_L(M) = \{x \in L : [M, x] = 0\}$, is an open ideal of L.*

Proof. The fact that $C_L(M)$ is an ideal follows immediately from axiom (iii) for L-modules. Let $a \in M$. Then $\phi : L \to M : x \mapsto [a, x]$ is continuous. Now, since the set $\{0\}$ is open in M, it follows that its pre-image under ϕ, $C_L(a) = \{x \in L : [a, x] = 0\}$, is open in L. Therefore $C_L(M) = \bigcap_{a \in M} C_L(a)$ is open as required. □

Lemma 3.1.5. *Let $L = \langle x_1, \ldots, x_s \rangle$ be a finitely generated Lie ring, and let $M = \langle a_1, \ldots, a_t \rangle_L$ be a finitely generated L-module. If a submodule N has index m in M then N is $(m(m + s) + t)$-generated as an L-module.*

Proof. Fix $e_1 + N, \ldots, e_m + N$ such that $M/N = \{e_1 + N, \ldots, e_m + N\}$. There exist functions β, γ, δ and elements $\{b_i \in N : 1 \leq i \leq t\}$, $\{c_{ij} \in N : 1 \leq i, j \leq m\}$,

$\{d_{ij} \in N : 1 \leq i \leq m, 1 \leq j \leq s\}$, such that, for each i, j, we have

$$a_i = e_{\beta(i)} + b_i$$

$$e_i + e_j = e_{\gamma(i,j)} + c_{ij}$$

$$[e_i, x_j] = e_{\delta(i,j)} + d_{ij}.$$

Let R be the L-submodule of N generated by all the b_i, c_{ij}, and d_{ij}. Now consider $S = \bigcup_{i=1}^{m}(e_i + R) \subseteq M$. We claim that $S = M$. Clearly each of the generators a_i of M are in S. Furthermore S is an L-module since

$$(e_i + r) + (e_j + r') = (e_i + e_j) + (r + r') = e_{\gamma(i,j)} + (c_{ij} + r + r') \in S$$

and

$$[e_i + r, x_j] = [e_i, x_j] + [r, x_j] = e_{\delta(i,j)} + (d_{ij} + [r, x_j]) \in S;$$

hence, $S = M$, as claimed.

Now, consider the surjective map $M/R \to M/N$. Because $|M/R| = |S/R| \leq m$, while $|M/N| = m$, the map is also injective. Thus $R = N$, and so N is $(m(m+s)+t)$-generated, as claimed. □

Proposition 3.1.6. *Let $L = \overline{\langle x_1, \ldots, x_s \rangle}$ be a finitely generated pro-finite Lie ring, let $M = \overline{\langle a_1, \ldots, a_t \rangle_L}$ be a finitely generated pro-finite L-module, and let N be an open submodule of M. If N has finite index m in M, then N is a $(m(m+s)+t)$-generated pro-finite L-submodule of M.*

Proof. First consider the case when M is finite. By Lemma 3.1.4, $C_L(M)$ is an open ideal, and hence $L/C_L(M)$ is finite. Therefore, by Lemma 3.1.5, N is $(m(m+s)+t)$-

generated as an $L/C_L(M)$-module, and since the action of $L/C_L(M)$ on M is identical to that of L, it follows that N is $(m(m+s)+t)$-generated as an L-module.

Now consider the general case when M may be infinite. Fix $K \leq_o M$. Then $(N+K)/K \leq M/K$ and M/K is a finite nilpotent L-module with $|(M/K)/((N+K)/K)| = m' \leq m$. Thus, by the previous case, $(N+K)/K$ is $(m'(m'+s)+t) \leq (m(m+s)+t)$ generated as an L-module. Now let $r = m(m+s)+t$ and let Y_K be the (finite) set of all r-tuples of generators of $(N+K)/K$. Then $\{Y_K : K \leq_o M\}$ can be viewed as an inverse system of compact spaces. So, by Tychonoff's theorem, $\varprojlim_{K \leq_o M} Y_K$ is non-empty; that is, there exists $n_1, \ldots, n_r \in N$ such that $n_1 + K, \ldots, n_r + K$ generate $(N+K)/K$ for all $K \leq_o M$. Thus, by part (iii) of Lemma 3.1.1,

$$N = \bigcap_{K \leq_o M} (\langle n_1, \ldots, n_r \rangle_L + K) = \overline{\langle n_1, \ldots, n_r \rangle_L},$$

and N is r-generated, as claimed. Finally N is pro-finite by part (i) of Lemma 3.1.1. □

Notice, by part ((ii)) of Lemma 3.1.1, if M in the statement of Proposition 3.1.6 is pro-nilpotent, then so is N.

Theorem 3.1.7. *Let L be a finitely generated pro-finite Lie ring, and let M be a finitely generated pro-nilpotent L-module. If a submodule N has finite index in M then N is open in M.*

Proof. Let m denote the index of N in M, and consider $P = mM + \overline{[M, L]}$. We claim that P is a L-submodule of M. Fix $K \leq_o M$. Then $mM + [M, L] + K/K$ is an L-submodule of M/K. It follows that $[mM+[M,L]+K,L] \subseteq mM+[M,L]+K$, and hence

$$[P,L] \subseteq \bigcap_{K \leq_o M} (mM+[M,L]+K) = \overline{mM+[M,L]} = mM+\overline{[M,L]},$$

proving the claim.

Now, since P is closed and of finite index in M, it follows that P, and hence $N+P$, is open in M. Furthermore, since $mM \subseteq N$ and, by Lemma 3.1.3, $\overline{[M,L]} \subseteq \Phi_L(M)$, it follows that $N+P \subseteq N+\Phi_L(M)$. Next, observe that, because N is a proper submodule of M, by Lemma 3.1.2, so is $N+\Phi_L(M)$. Therefore

$$N \leq N+P \leq N+\Phi_L(M) < M,$$

and so the index of N in $N+P$ is strictly less than m. Now, by Proposition 3.1.6, $N+P$ is a finitely generated pro-nilpotent L-submodule of M. Thus, by induction on the index of N in M, we may assume that N is open in $N+P$. However, since $N+P$ was open in M, it follows that N is open in M. □

Since a pro-nilpotent Lie ring L is clearly pro-nilpotent as an L-module. Theorem B is a straightforward corollary of Theorem 3.1.7.

3.2 Related results

After proving that every subgroup of finite index in a finitely generated pro-soluble group is open, Segal ([21]) deduced that each term of its lower central series is closed. The fact that each term of the lower central series of an arbitrary finitely generated pro-finite Lie ring is closed is relatively simple:

Theorem C. *Let* $L = \overline{\langle x_1, \dots, x_t \rangle}$ *be a finitely generated pro-finite Lie ring, and let* M, N *be closed ideals of* L *with* $M = \overline{\langle a_1, \dots, a_s \rangle_L}$ *finitely generated as a pro-finite* L*-module. Then the ideal* $[M, N]$ *of* L *is closed.*

Proof. Take $X = \sum_{i=1}^{s} [a_i, N]$ and $Y = \sum_{j=1}^{t} [x_j, M, N]$ and consider the maps

$$\phi : \underbrace{N \times \cdots \times N}_{s} \to L$$

$$\phi(n_1, \dots, n_s) = \sum_{i=1}^{s} [a_i, n_i]$$

$$\psi : \underbrace{M \times \cdots \times M}_{t} \times \underbrace{N \times \cdots \times N}_{t} \to L$$

$$\psi(m_1, \dots, m_t, n_1, \dots, n_t) = \sum_{j=1}^{t} [x_j, m_j, n_j].$$

Clearly ϕ and ψ are continuous maps from a compact space to a Hausdorff space, thus the images $\mathrm{Im}(\phi) = X$ and $\mathrm{Im}(\psi) = Y$ are both closed.

Let $K \lhd_o L$. Then

$$\begin{aligned} M &\subseteq \langle a_1, \dots, a_s \rangle_L + K \\ &= \langle a_1, \dots, a_s \rangle_+ + \sum_{j=1}^{t} [x_j, M] + K, \end{aligned}$$

and so

$$[M, N] \subseteq \sum_{i=1}^{s} [a_i, N] + \sum_{j=1}^{t} [x_j, M, N] + [K, N] \subseteq X + Y + K$$

Therefore we have

$$[M,N] \subseteq \bigcap_{K \lhd_O L} (X+Y+K) = \overline{X+Y} = X+Y \subseteq [M,N],$$

and hence $[M,N] = X+Y$ is closed. □

Using induction on n it is easy to see that if L is finitely generated then each $\gamma_n(L)$ is a finitely generated pro-finite L-module. Thus we have the following corollary.

Corollary 3.2.1. *Let L be a finitely generated pro-finite Lie ring, then:*

(i) *$\gamma_n(L)$ is closed for all $n \geq 1$;*

(ii) *$[\gamma_{n_1}(L), \ldots, \gamma_{n_s}(L)]$ is closed for all $n_1, \ldots, n_s \geq 1$; and,*

(iii) *if I is an ideal such that L/I is finite and nilpotent, then I is open in L.*

We remark that, *a priori* to Theorem B, it is not at all clear that all finite images of a finitely generated pro-nilpotent Lie ring are nilpotent.

Chapter 4

The Kurosh Problem for pro-finite Lie rings

It is known that Engel's theorem holds for finite Lie rings; that is, every finite Lie ring is nilpotent when each of its elements is ad-nilpotent (see, for example, Theorem 1.7.2 of [2]). Recall that an element y in L is said to be ad-nilpotent if there exists a positive integer n such that

$$[x,_n y] = (x)(\text{ad } y)^n = 0,$$

for all x in L. The simplest form of the Kurosh Problem for Lie rings asks whether or not this result generalises to finitely generated Lie rings; in other words, must a Lie ring be locally finite if each of its elements is ad-nilpotent? However, for every prime p, Golod constructed in [5] an example of an infinite finitely generated Lie ring of characteristic p with every element ad-nilpotent.

Consequently, it makes sense to restrict the hypothesis in Kurosh's problem. To this end, we intend to give a positive solution of the Kurosh Problem for all pro-finite Lie rings. In fact, we shall prove slightly more. We shall call a Lie ring L *Engelian* if, for every $x, y \in L$, there exists some $n = n(x, y)$ such that $[x,_n y] = 0$.

Theorem D. *Engelian pro-finite Lie rings are locally nilpotent.*

Recall that a topological Lie ring L is called *locally nilpotent* if each of its finitely generated closed subrings is nilpotent.

We remark that Theorem D is equivalent to the statement that every finitely generated Engelian pro-finite Lie ring is finite.

Our proof uses techniques derived from those used by Wilson and Zelmanov to prove that all periodic pro-finite groups are locally finite ([28] and [31]) and that all Engelian pro-finite groups are locally nilpotent ([30]). Interestingly, these techniques were essentially Lie-theoretic in nature to start.

In the final section, we address the Kurosh problem for pro-finite restricted Lie algebras.

4.1 Proof of Theorem D

First, we introduce some more notation. Given any field $\mathbb{F}$, let $A_n(\mathbb{F})$ denote the free Lie algebra over $\mathbb{F}$ on n generators, and let $A_\infty(\mathbb{F})$ denote the free Lie algebra over $\mathbb{F}$ on a countable set of generators. Let $f \in A_n(\mathbb{F})$. Recall that a Lie $\mathbb{F}$-algebra L is said to satisfy the polynomial identity $f = 0$ if $f(y_1, \ldots, y_n) = 0$ for all $y_1, \ldots, y_n \in L$. Extending this notion, we shall say that cosets $x_1+I, \ldots, x_n+I$ of an ideal I of L satisfy the coset identity $f = 0$ if $f(x_1 + y_1, \ldots, x_n + y_n) = 0$ for all $y_1, \ldots, y_n \in I$.

It is well-known that the $\mathbb{F}$-space

$$\mathrm{gr}(L) = \bigoplus_{i \geq 1} \gamma_i(L)/\gamma_{i+1}(L)$$

admits the structure of a Lie $\mathbb{F}$-algebra induced by the product of homogeneous elements given by

$$[x+\gamma_{i+1}(L), y+\gamma_{j+1}(L)] = [x,y]+\gamma_{i+j+1}(L).$$

Theorem 4.1.1. *Let L be a Lie algebra over a field $\mathbb{F}$, and let $f \in A_\infty(\mathbb{F})$ be non-trivial. If there exists $a_1,\dots,a_n \in L$ and $I \lhd L$ of finite index in L such that $f=0$ is satisfied on the cosets $a_1+I,\dots,a_n+I$ then* $\mathrm{gr}(L)$ *satisfies a non-trivial polynomial identity.*

Proof. Without loss of generality, we can assume that L is residually nilpotent and f is a polynomial expressed as a linear combination of basic commutators with no commutators of degree less than some minimal $m \geq 2$ (we can replace f by $[f, x_1]$ if required). Choose a multi-degree $(d_1,\dots,d_n)$ such that $\sum_{i=1}^n d_i = m$ and f has a non-zero component of this multi-degree. Now, consider the polynomial f_1, in the free Lie algebra on $\{x_1,\dots,x_n,z_1,\dots,z_m\}$, defined by

$$f_1(x_1,\dots,x_n,z_1,\dots,z_m) = f(x_1+z_1+\cdots+z_{d_1},\dots,x_n+z_{d_{n-1}+1}+\cdots+z_m).$$

Note that all commutators in the linear expansion of f_1 are of degree at least m.

We may express f_1 as

$$f_1 = \sum_{j=1}^{m+1} \sigma_j$$

such that, for each $1 \leq j \leq m+1$, σ_j is the sum of commutators in f_1 that do not involve z_j, but involve all of z_i for $i<j$, and σ_{m+1} is the sum of commutators in

f_1 that involve all of $z_1, \ldots, z_m$. For each $1 \leq i \leq m+1$, define polynomials

$$f_i = \sum_{j=i}^{m+1} \sigma_j.$$

We claim that, for any $b_1, \ldots, b_m \in I$, we have $f_i(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ for each $1 \leq i \leq m+1$. We proceed by induction. For any $b_1, \ldots, b_m \in I$, we have

$$\begin{aligned} &f_1(a_1, \ldots, a_n, b_1, \ldots, b_m) \\ &\quad = f(a_1 + b_1 + \cdots + b_{d_1}, \ldots, a_n + b_{d_{n-1}+1} + \cdots + b_m) \\ &\quad = 0. \end{aligned}$$

We may assume $f_i(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ and consider f_{i+1}. Now, since f_{i+1} is a sum of commutators that all involve z_i, we have

$$f_{i+1}(a_1, \ldots, a_n, b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_m) = 0.$$

So,

$$\begin{aligned} &\sigma_i(a_1, \ldots, a_n, b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_m) \\ &\quad = \sigma_i(a_1, \ldots, a_n, b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_m) \\ &\qquad + f_{i+1}(a_1, \ldots, a_n, b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_m) \\ &\quad = f_i(a, \ldots, a_n, b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_m) = 0 \end{aligned}$$

However no commutators in σ_i involve z_i, and hence $\sigma_i(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$.

Thus

$$\begin{aligned} f_{i+1}(a_1, \ldots, a_n, b_1, \ldots, b_m) &= \sigma_i(a_1, \ldots, a_n, b_1, \ldots, b_m) \\ &\quad + f_{i+1}(a_1, \ldots, a_n, b_1, \ldots, b_m) \\ &= f_i(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0, \end{aligned}$$

completing the induction.

In particular, we have $f_{m+1}(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, for all $b_1, \ldots, b_m \in I$. Now write $f_{m+1} = g + h$ such that g is the sum of all commutators in f_{m+1} of degree m and h is the sum of commutators of degree greater than m. Note that g is the (non-trivial) multi-linearization of the homogeneous component of f with multi-degree $(d_1, \ldots, d_m)$. Since g does not involve any of $x_1, \ldots, x_n$, for clarity we will write $g = g(z_1, \ldots, z_m)$ from here on.

For each $x \in L$, define $\nu(x)$ to be the integer k such that $x \in \gamma_k(L) \setminus \gamma_{k+1}(L)$. Because

$$g(b_1, \ldots, b_m) + h(a_1, \ldots, a_n, b_1, \ldots, b_m) = f_{m+1}(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$$

and all commutators in h are of degree at least $m+1$ and involve all of $z_1, \ldots, z_m$, it follows that

$$\nu(g(b_1, \ldots, b_m)) = \nu(h(a_1, \ldots, a_n, b_1, \ldots, b_m) \geq 1 + \sum_{i=1}^{m} \nu(b_i).$$

Next, observe that, since I has finite index in L, the lower central series of L/I stabilises; that is there exists some integer t such that $\gamma_t(L) \subseteq I + \gamma_k(L)$ for all

$k \geq t$. Let $r_{ij} \in L$ for $1 \leq i \leq m$, $1 \leq j \leq t$, and set $e_i = [r_{i1}, \ldots, r_{it}]$ for each i. Then each $e_i \in \gamma_t(L) \subseteq I + \gamma_k(L)$ for all $k \geq t$, and

$$\nu(e_i) \geq \sum_j \nu(r_{ij}).$$

Therefore, for each i, there exists $b_i \in I$ and $c_i \in L$, such that $e_i = b_i + c_i$ and $\nu(c_i) > \nu(e_i)$. Notice that then $\nu(b_i) = \nu(e_i)$, and so $\nu(c_i) > \nu(b_i)$. Hence

$$\begin{aligned} \nu(g(e_1, \ldots, e_m)) &= \nu(g(b_1, \ldots, b_m)) \\ &\geq 1 + \sum_{i=1}^{m} \nu(b_i) \\ &= 1 + \sum_{i=1}^{m} \nu(e_i) \\ &\geq 1 + \sum_{i,j} \nu(r_{ij}) \end{aligned}$$

Now consider the element $\hat{g}(x_{11}, \ldots, x_{mt}) = g([x_{11}, \ldots, x_{1t}], \ldots, [x_{m1}, \ldots, x_{mt}])$ in the free Lie $\mathbb{F}$-algebra on $\{x_{11}, \ldots, x_{mt}\}$. From above, we see that for any homogeneous $r_{11}, \ldots, r_{mt} \in \mathrm{gr}(L)$ we have

$$\hat{g}(r_{11}, \ldots, r_{mt}) = 0.$$

Consequently, $\mathrm{gr}(L)$ satisfies the non-trivial multilinear polynomial identity $\hat{g} = 0$, as required. □

For each prime p, let $\mathbb{F}_p$ denote the field of p elements.

Corollary 4.1.2. *Let L be a pro-finite Lie ring of prime characteristic $p > 0$.*

Then either $A_2(\mathbb{F}_p)$ embeds in L or the Lie $\mathbb{F}_p$-algebra $\mathrm{gr}(L)$ satisfies a non-trivial polynomial identity.

Proof. For each non-zero $f \in A_2(\mathbb{F}_p)$, write $S_f = \{(a,b) \in L \times L | f(a,b) = 0\}$. Clearly each S_f is closed in $L \times L$. If $A_2(\mathbb{F}_p)$ cannot be embedded in L, then we must have

$$L \times L = \bigcup_{f \in A_2(\mathbb{F}_p) \setminus \{0\}} S_f.$$

Thus, by the Baire Category Theorem, there exists some $f \in A_2(\mathbb{F}_p)$, $I \lhd_o L$, and $a, b \in L$, such that $f(a+I, b+I) = 0$. The result now follows from Theorem 4.1.1. □

It may be interesting to note that this corollary fails for arbitrary Lie $\mathbb{F}_p$-algebras, even for those that are finitely generated. Indeed, there exist counterexamples L to the Kurosh Problem such that L is isomorphic to $\mathrm{gr}(L)$ (see [14]). Since L is Engelian, certainly L cannot contain $A_2(\mathbb{F}_p)$; however, neither can $L \cong \mathrm{gr}(L)$ satisfy a non-trivial polynomial identity as can be seen by the next result. This deep theorem is the fundamental basis of our proof.

Zelmanov's Theorem. *([32]) Let L be a finitely generated Lie algebra over a field of positive characteristic. If every commutator (of length one or more) in the generators of L is ad-nilpotent, and L satisfies a non-trivial polynomial identity, then L is nilpotent.*

A Lie ring L is called *n-Engelian* if L satisfies the polynomial identity $[x,_n y] = 0$; L is *bounded Engelian* if L is n-Engelian for some positive integer n. It follows from Zelmanov's Theorem that every bounded Engelian Lie $\mathbb{F}_p$-algebra is locally nilpotent. In fact, according to Zelmanov, the following more general corollary can also be deduced.

Corollary 4.1.3. *Every bounded Engelian Lie ring is locally nilpotent.*

Lemma 4.1.4. *Let L be any pro-finite Lie ring. If L is Engelian then every element in L is ad-nilpotent.*

Proof. By assumption, for any $a, b \in L$, there exists some $n = n(a, b)$ such that $[a,_n b] = 0$. Fix b and consider sets $T_n = \{a \in L | [a,_n b] = 0\}$. Clearly each T_n is closed, and $\bigcup_{n \geq 1} T_n = L$. By the Baire Category Theorem, there must exist some $a \in L$ and $I \lhd_o L$, such that $a + I \subseteq T_{n_0}$ for some n_0. That is, $[a + y,_{n_0} b] = 0$ for all $y \in I$. Now, as L/I is finite, there exists some $a_1, \ldots, a_m \in L$ such that, for any $x \in L$, we have $x = a_i + y$ for some $y \in I$. Next, choose $n \geq n_0$ such that $[a_i - a,_n b] = 0$ for all $1 \leq i \leq m$. Then, for every $x \in L$, we have

$$[x,_n b] = [a_i + y,_n b] = [a_i - a,_n b] + [a + y,_n b] = 0.$$

This proves $(\text{ad } b)^n = 0$, and so every element in L is ad-nilpotent. □

Proposition 4.1.5. *Let L be a finitely generated pro-finite Lie ring of prime characteristic $p > 0$. If L is Engelian then L is nilpotent.*

Proof. It follows from Engel's theorem that L is residually nilpotent; consequently, it suffices to prove that $\text{gr}(L)$ is nilpotent. To accomplish this, we need only show that $\text{gr}(L)$ satisfies the hypotheses of Zelmanov's Theorem. Clearly $\text{gr}(L)$ is a Lie algebra over $\mathbb{F}_p$. Since L is finitely generated, by Proposition 2.1.3 of Chapter 2, the Frattini subring $\Phi(L) = pL + \gamma_2(L) = \gamma_2(L)$ is open in L, and so $\text{gr}(L)$

is generated by the finite set $\gamma_1(L)/\gamma_2(L)$. By Lemma 4.1.4, every element of L is ad-nilpotent; hence, the homogeneous elements of $\mathrm{gr}(L)$ are also ad-nilpotent. Finally, since $A_2(\mathbb{F}_p)$ cannot embed in L, $\mathrm{gr}(L)$ satisfies a non-trivial polynomial identity by Corollary 4.1.2, as required. □

Proposition 4.1.6. *Let L be a finitely generated pro-p Lie ring. If L is Engelian then L is nilpotent.*

Proof. From Proposition 4.1.5, applied to the pro-finite quotient L/pL, it follows that L/pL is nilpotent. Because L is finitely generated, it is readily deduced that L/pL is finite. Thus, since pL is closed, it is also open. This allows us to employ the Theorem A, part (viii), which informs us of the existence of a positive integer r such that the ideal $U = p^r L$ is both open in L and free as a $\mathbb{Z}_p$-module on a finite basis $\{x_1, \ldots, x_d\}$, say.

We claim U is nilpotent. To this end, notice that for any $\tilde{y} \in \tilde{U} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U$, we can write

$$\tilde{y} = \sum_{i=1}^{d} \frac{\alpha_i}{\beta_i} \otimes_{\mathbb{Z}_p} x_i,$$

for some $\alpha_i, \beta_i \in \mathbb{Z}_p$. Then $\beta\tilde{y} \in U$, where $\beta = \prod_{i=1}^{d} \beta_i \in \mathbb{Z}_p$. By Lemma 4.1.4, it follows that there exists some integer n such that

$$\beta^n [x_{i,n}\, \tilde{y}] = [x_{i,n}\, \beta\tilde{y}] = 0,$$

for each i. Since the $\mathbb{Z}_p$-action on $\tilde{U}$ is free, we may conclude $(\mathrm{ad}\ \tilde{y})^n = 0$. It now follows from Engel's theorem that the finite-dimensional Lie $\mathbb{Q}_p$-algebra $\tilde{U}$ is nilpotent. Consequently, U is nilpotent, as well.

Let m be the nilpotence class of U. Then $p^{rm}L$ is contained in the m^{th} upper centre of L since

$$[(p^r)^m L,_m L] = \gamma_{m+1}(p^r L) = 0.$$

But $p^{rm}L$ is open in L by Theorem A, part (v), and so $L/p^{rm}L$ is nilpotent of class n, say. It follows at once that L is nilpotent of class at most $m+n$. □

A *supernatural number* is a formal infinite product $\prod p^{n(p)}$ over all primes p, where each $n(p)$ is non-negative, and possibly infinite. Divisibility can be defined in a natural way: $\prod p^{m(p)}$ divides $\prod p^{n(p)}$ if $m(p) \leq n(p)$ for all primes p. The lowest common multiple of a family $\{\prod p^{n(p,\alpha)} | \alpha \in I\}$ of supernatural numbers is

$$\underset{\alpha \in I}{\mathrm{LCM}} \left\{ \prod p^{n(p,\alpha)} \right\} = \prod p^{s(p)},$$

where $s(p) = \sup_{\alpha \in I}\{n(p,\alpha)\}$. Given a profinite group G, with closed subgroup H, we can then define $|G : H|$ to be the supernatural number

$$\underset{N \triangleleft_O G}{\mathrm{LCM}} \left\{ |G : NH| \right\},$$

and $|G| = |G : 1|$.

If G is a pro-finite group and p is a prime, then H is a Sylow p-subgroup of G if $|H|$ is a (possibly infinite) power of p, and $|G : H|$ is co-prime to p. The Sylow theorems extend to pro-finite groups using supernatural numbers (see [29] for a detailed account). In particular, if G is a pro-nilpotent group then G is isomorphic to the Cartesian product of its Sylow subgroups.

Let L be a pro-nilpotent Lie ring. We have a group isomorphism

$$\phi : (L, +) \to \prod_{p \text{ prime}} K_p,$$

where K_p is the Sylow p-subgroup of $(L, +)$. We claim that, for each prime p, K_p is a closed ideal of L, and hence ϕ is a Lie ring isomorphism. Since, for each prime p, K_p is a closed subgroup of $(L, +)$ it is closed in L, so it remains only to show that K_p is an ideal. Fix a prime p nd consider the Abelian subgroup $H = [K_p, L]$ of $(L, +)$. Since K_p is a Sylow p-subgroup, for any ideal $I \lhd_o L$ there exists some $n \geq 1$ such that $p^n K_p \subseteq I$. Thus for any $I \lhd_o L$ we can find $n \geq 1$ such that

$$p^n H = [p^n K_p, L] \subseteq [I, L] \subseteq I.$$

Since the open ideals of L form a neighbourhood base of 0 it follows that $|H| = \text{LCM}_{I \lhd_o H} |H : I|$ is a (possibly infinite) power of p. Now, since $|L : K_p|$ is co-prime to p, it follows that $H \subseteq K_p$, and thus K_p is an ideal as claimed. Furthermore, each K_p is a pro-p Lie ring since each K_p is closed and hence pro-nilpotent and is additively pro-p as an Abelian group.

We are now ready to prove Theorem D. Let L be a finitely generated Engelian pro-finite Lie ring. It follows from Lemma 4.1.4 and Engel's theorem for finite Lie rings (see Theorem 1.7.2 of [2]), that L is pro-nilpotent. Let $T_n = \{b \in L | (\text{ad } b)^n = 0\}$. Then each T_n is closed and $L = \bigcup_{n \geq 1} T_n$, so by the Baire Category Theorem there exists $n \geq 1, b \in L$, and $I \lhd_o L$ such that $[x,_n b + I] = 0$, for all $x \in L$. Let $p_1^{e_1} \cdots p_r^{e_r}$ be the prime decomposition of $m = |L/I|$, and let $P_1, \ldots, P_r$ be Sylow ideals of L corresponding to the primes $p_1, \ldots, p_r$. Set $K = P_1 + \cdots + P_r$. Then, as pro-finite Lie rings, we have

$$L \cong P_1 \times \cdots \times P_r \times (L/K).$$

We claim that L/K is n-Engelian. First observe that $mb \in K$, and so $m[x,_n y] = m[x,_n b + y] \in K$ for all $x \in L$ and $y \in I$. Since all elements of $L/K = I + K/K$ have additive order co-prime to m, the claim follows. Thus, L/K is a finitely generated n-Engelian pro-finite Lie ring. It now follows from Corollary 4.1.3 that L/K has a dense nilpotent subring; consequently, L/K itself is nilpotent. Finally, to complete our proof, we apply Proposition 4.1.6 to each of the P_i.

4.2 Restricted Lie algebras

The main result of this chapter can be extended to the class of restricted Lie algebras. Recall that a Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$ is called a *restricted Lie algebra* if it has an additional operation $x \mapsto x^p$, called a *p-map*, satisfying the following capability axioms:

(i) $(\lambda x)^p = \lambda^p x^p$ when $\lambda \in \mathbb{F}, x \in L$;

(ii) $\mathrm{ad}(x^p) = (\mathrm{ad}\ x)^p$; and,

(iii) $(x + y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y)$; where $is_i(x, y)$ is the coefficient of t^{i-1} in the polynomial $(x)\,\mathrm{ad}(tx + y)^{p-1} \in L[t]$.

A subalgebra K of L is called a *p-subalgebra* if K is closed under the p-map. A set X generates L if L is the smallest p-subalgebra containing X. We shall say L

is *p-nil* if, for all $x \in L$, there exists $t = t(x) \geq 1$ such that $x^{p^t} = 0$; if t can be chosen independently of x then L is called *p-nilpotent.*

Following [17], we shall call a restricted Lie algebra *finite-p* if it is finite and p-nilpotent. We remark that axiom (ii) above together with Engel's theorem imply that all finite-p restricted Lie algebras are nilpotent. A *pro-p restricted Lie algebra* is a pro-finite restricted Lie algebra L with the property that the ideals I, with L/I finite-p, form a neighbourhood base of 0.

Theorem E. *Let L be a finitely generated pro-finite restricted Lie $\mathbb{F}_p$-algebra. Then the following statements hold.*

(i) *If L is p-nil then L is finite-p.*

(ii) *If L is Engelian then L is nilpotent. Furthermore, in this case, the centre of L is open.*

Before proving Theorem E, we require a bit more notation. Let

$$\mathrm{gr}_p(L) = \bigoplus_{n \geq 1} \gamma_n(L) + D_{n+1}(L)/D_{n+1}(L),$$

where $D_{n+1}(L)$ denotes the p-ideal

$$D_{n+1}(L) = \sum_{ip^j \geq n+1} \gamma_i(L)^{p^j},$$

for each n. It is easy to see that $\mathrm{gr}_p(L)$ can be viewed as a Lie $\mathbb{F}_p$-algebra in such a way that $\mathrm{gr}_p(L)$ is an epimorphic image of $\mathrm{gr}(L)$.

Proof. First, note that if either L is p-nil or Engelian then $\mathrm{gr}(L)$ satisfies a non-trivial polynomial identity by Corollary 4.1.2. Therefore, since $\mathrm{gr}(L)$ satisfies a non-trivial polynomial identity, so must its image, $\mathrm{gr}_p(L)$.

Now, to prove (i), suppose that L is p-nil. Then L is a finitely generated pro-p restricted Lie $\mathbb{F}_p$-algebra. Consequently, by [17], $D_2(L)$ is open in L, and so $\mathrm{gr}_p(L)$ is generated as a Lie $\mathbb{F}_p$-algebra by the finite component $L/D_2(L)$. Furthermore, since L is p-nil, it follows from axiom (ii) above that every element of L is ad-nilpotent, and hence all homogeneous elements of $\mathrm{gr}_p(L)$ are ad-nilpotent. By Zelmanov's Theorem, it follows that $\mathrm{gr}_p(L)$ is nilpotent; that is, there exists $m \geq 1$ such that $\gamma_m(L) \subseteq D_{m+1}(L)$. But then $\gamma_{m+1}(L) = \gamma_{m+2}(L)$ (see [17]), and thus L is nilpotent of class at most m.

Let $X = \{x_1, \ldots, x_d\}$ be a (topological) generating set of L, let $A \subseteq L$ be the Lie $\mathbb{F}_p$-algebra generated by X, and let A_p be the p-subalgebra generated by A. Since L is nilpotent, there exist $y_1, \ldots, y_m \in A$ such that $A = \mathbb{F}_p y_1 + \cdots + \mathbb{F}_p y_m$. Then, by Proposition 2.1.3 of [25], $A_p = \sum_{i,j} \mathbb{F}_p {y_i}^{p^j}$ is finite. However, A_p is dense in L, and so L is finite-p.

To prove (ii), suppose that L is Engelian. Then, by Lemma 4.1.4, every element of L is ad-nilpotent. Now, since the centre of L, $Z(L)$, is closed, it follows that ad $L \cong L/Z(L)$ is a finitely generated p-nil pro-p restricted Lie $\mathbb{F}_p$-algebra. By part (i), $\mathrm{ad}(L)$ is therefore finite; in other words, the closed p-ideal $Z(L)$ is open. But then $L/Z(L)$ is finite-p. Consequently L is nilpotent, too, as required. □

Chapter 5

Pro-p Lie rings of infinite Prüfer rank

In [23] Shalev proved that every finitely generated pro-p group of finite Prüfer rank cannot involve all wreath products of the form $C_p \wr C_{p^n}$, $n \geq 1$, as closed sections. Shalev went on to pose the still open problem of whether a finitely generated pro-p group G has finite Prüfer rank if and only if it does not involve $C_p \wr \mathbb{Z}_p = \varprojlim_{n\geq 1} C_p \wr C_{p^n}$ as a closed section. In Chapter 2 we proved a result analogous to that of Shalev, namely that every finitely generated pro-p Lie ring of finite Prüfer rank cannot involve $W_n(\mathbb{F}_p) = \langle e_{12}, te_{22}\rangle \subseteq \mathfrak{gl}_2(\mathbb{F}_p[t]/\langle t^n\rangle)$, for all $n \geq 1$, as closed sections. A natural question arises:

> *Does a finitely generated pro-p Lie ring L have finite Prüfer rank if and only if it does not involve $W_\infty(\mathbb{F}_p) = \varprojlim_{n\geq 1} W_n(\mathbb{F}_p)$ as a closed section?*

We remark that the Lie ring $W_\infty(\mathbb{F}_p)$ can be realised as the Lie $\mathbb{F}_p$-algebra $\overline{\langle e_{12}, te_{22}\rangle} = \mathbb{F}_p[[t]]e_{12} + \mathbb{F}_p te_{22} \subseteq \mathfrak{gl}_2(\mathbb{F}_p[[t]])$.

We seek to draw together the results of the preceeding three chapters to prove a slightly weaker result, involving the derived series. The *derived series* of a pro-p Lie ring L is defined by $\delta_1(L) = L$ and $\delta_n(L) = [\delta_{n-1}(L), \delta_{n-1}(L)]$. Our main result is as follows:

Theorem F. *Let L be a finitely generated pro-p Lie ring. If there exists a positive integer n such that $\delta_n(L) \subseteq pL$ then L has finite Prüfer rank if and only if it does not involve $W_\infty(\mathbb{F}_p)$ as a closed section. In particular, if L is soluble then L has finite Prüfer rank if and only if it does not involve $W_\infty(\mathbb{F}_p)$ as a closed section.*

5.1 Proof of Theorem F

We first consider the restricted case of pro-p Lie rings with characteristic p:

Proposition 5.1.1. *Let L be a finitely generated pro-p Lie ring with $pL = 0$. If L does not involve $W_\infty(\mathbb{F}_p)$ as a closed section then the ideal $\delta_n(L)$ is open in L for each $n \geq 1$.*

Proof. Suppose that L does not involve $W_\infty(\mathbb{F}_p)$ as a closed section. By Theorem B it will suffice to show that $|L/\delta_n(L)|$ is finite for each $n \geq 1$. We will proceed by induction on n. The case $n = 1$ is trivial, and for $n = 2$ we have $|L/\delta_2(L)| = |L/\Phi(L)|$, which is finite since L is finitely generated. Now, suppose that there exists some $N \geq 1$ such that $|L/\delta_n(L)|$ is finite for each $n \leq N$. Since $\delta_{N-1}(L)$ has finite index in L it follows, by Theorem B, that $\delta_{N-1}(L)$ is open (and hence closed) in L, and so $\delta_{N-1}(L)$ is pro-p. Therefore $\delta_{N-1}(L)$ is finitely generated since

$$|\delta_{N-1}/\Phi(\delta_{N-1}(L))| = |\delta_{N-1}(L)/\delta_N(L)|$$

which is finite by hypothesis. Now, by Corollary 3.2.1,

$$\delta_{N+1}(L)) = [\gamma_2(\delta_{N-1}(L)), \gamma_2(\delta_{N-1}(L))]$$

is closed in $\delta_{N-1}(L) \lhd_o L$, and thus closed in L. So $\delta_{N-1}(L)/\delta_{N+1}(L)$ is a finitely generated metabelian closed section of L. Furthermore, $W_\infty(\mathbb{F}_p)$ is not involved in $\delta_{N-1}(L)/\delta_{N+1}(L)$ since it is not involved in L. We claim that $\delta_{N-1}(L)/\delta_{N+1}(L)$ is nilpotent, and hence finite. Suppose, to the contrary, that $\delta_{N-1}(L)/\delta_{N+1}(L)$ is not nilpotent. Then, by Theorem D, $\delta_{N-1}(L)/\delta_{N+1}(L)$ is not Engelian, and there exists $x, y \in \delta_{N-1}(L)/\delta_{N+1}(L)$ such that $[x,_m y] \neq 0$ for all $m \geq 1$. But then $\overline{\langle x, y\rangle}$ is isomorphic to $W_\infty(\mathbb{F}_p)$, a contradiction. □

Corollary 5.1.2. *Let L be a finitely generated pro-p Lie ring. If L does not involve $W_\infty(\mathbb{F}_p)$ as a closed section then $\delta_n(L) + pL$ is open in L for arbitrarily large n.*

Proof. Let $L_p = L/pL$. Then L_p is a finitely generated pro-p Lie ring with $pL_p = 0$, and since $W_\infty(\mathbb{F}_p)$ is not involved in L, it is not involved in L_p either. Therefore, from the proposition, we have $\delta_n(L_p)$ open in L_p for each $n \geq 1$, and hence $\delta_n(L) + pL$ is open in L as claimed. □

The proof of Theorem F is now straightforward. Let L be a finitely generated pro-p Lie ring, and let N be a positive integer such that $\delta_N(L) \subseteq pL$. Clearly if L involves $W_\infty(\mathbb{F}_p)$ then L must involve $W_n(\mathbb{F}_p)$ for all $n \geq 1$, thus, by Theorem 2.3.4, L does not have finite Prüfer rank. Conversely, suppose that L does not involve $W_\infty(\mathbb{F}_p)$. Then by the previous corollary $\delta_n(L) + pL$ is open in L for all $n \geq 1$. However, since $\delta_N(L) \subseteq pL$, we have pL open in L and hence, by Theorem 2.3.4, L has finite Prüfer rank.

We remark that for a finitely generated pro-p Lie ring L, $pL+\delta_n(L)$ is open in L only if there exists $m \geq 1$ such that $\gamma_m(L) \subseteq \delta_n(L)$. Therefore, by Corollary 5.1.2, given a finitely generated pro-p Lie ring L with infinite Prüfer rank that does not involve $W_\infty(\mathbb{F}_p)$, there must exist a sequence of positive integers $m_1, m_2, m_3, \ldots$ such that $\gamma_{m_n} \subseteq \delta_n(L)$ for each $n \geq 1$. Whether such a pathological pro-p Lie ring exists is an open question.

Chapter 6

Open problems

Over the course of this work a number of questions have arisen that remain unresolved. This chapter highlights a number of the more significant such questions.

Problem 1. *The equivalence of* $\mathrm{rk}(L) < \infty$ *and* $pL \lhd_o L$ *for pro-p Lie rings leads us to propose that a similar relationship may hold in pro-p groups. We may frame the problem in terms of increasing specificity; does a pro-p group G have finite Prüfer rank if:*

(i) *the subset* $G^{\{p\}} = \{g^p | g \in G\}$ *is open in G;*

(ii) *the subset* $G^{\{p^n\}} = \{g^{p^n} | g \in G\}$ *is open for each* $n \geq 1$*;*

(iii) *the map* $G \to G : g \mapsto g^p$ *is open?*

We remark that the converses are true and that, for any finitely generated pro-p group, the subgroup G^p generated by $G^{\{p\}}$ is *always* open in G (see [4]).

Problem 2. *In Proposition 2.3.5 it was shown that a pro-p Lie ring L of finite Prüfer rank satisfies* $\mathrm{rk}(L) \leq d(L,+)$*. By Theorem 2.3.3, it follows that if L is powerful then* $\mathrm{rk}(L) = d(L,+)$*. Is the converse true?*

In [3] Barnea and Shalev showed that, for a pro-p group G, one can use the filtration $G_n = G^{p^n}$ to define a metric on G, and consequently the Hausdorff dimension for closed subgroups of G. Furthermore, it was shown that a finitely generated pro-p group G has finite Prüfer rank if and only if every infinite closed subgroup of G has non-zero Hausdorff dimension.

For the category of pro-p Lie rings, the analogous filtration $L_n = p^n L$ will not necessarily allow a metric to be defined if L does not have finite Prüfer rank. An alternative is to use the filtration provided by $L_n = D_n(L) = \prod_{ip^j \geq n} p^j \gamma_i(L)$. Under this filtration it can be shown that, if L has finite Prüfer rank, a closed subring $K \leq_c L$ will have zero Hausdorff dimension if and only if K is finite. Behaviour in the case of infinite Prüfer rank is not clear however:

Problem 3. *If L is a pro-p Lie ring with infinite Prüfer rank does L necessarily contain a closed subring with zero Hausdorff with respect to the filtration $L_n = D_n(L)$?*

From Chapter 3 we have the outstanding question:

Problem 4. *Is the topology of every finitely generated pro-finite Lie ring uniquely determined by its algebraic structure?*

Given the result of Chapter 3 and the work of Segal ([21]), it appears most profitable to first consider the case of finitely generated pro-soluble Lie rings.

Furthermore, in Chapter 3, it is only shown that ideals of finite index are open.

Problem 5. *Given a finitely generated pro-nilpotent Lie ring L, are all* subrings *of finite index in L necessarily open?*

We remark that, given the results of Chapter 3, Problem 5 is equivalent to the following:

Problem 6. *Given a finitely generated pro-nilpotent Lie ring L, and a subring $H \leq L$ of finite index, does there exist an ideal $I \lhd L$ of finite index such that $I \leq H \leq L$?*

In fact, we believe the following, more fundamental problem, to be open:

Problem 7. *Let L be an arbitrary (abstract) Lie ring, and let $H \leq L$ be a subring of finite index. Does H necessarily contain an ideal of L of finite index?*

We remark that this is obvious in the category of groups and was proved for associative rings by Hirano in [7]. We also note that Problem 7 fails for Lie algebras over fields of characteristic zero (see Example 2.2 in [18]).

The results of Chapter 4 also offer potential to be further generalised:

Problem 8. *Does the Kurosh Problem have a positive solution for compact Hausdorff Lie rings?*

In particular, if it could be shown that finitely generated compact Hausdorff Engelian Lie $\mathbb{F}_p$-algebras are necessarily residually nilpotent, we could develop a result analogous to Proposition 4.1.5 as follows: Let $L = \overline{\langle x_1, \ldots, x_d \rangle}$ be a finitely generated compact Hausdorff Lie $\mathbb{F}_p$-algebra, and suppose that L is Engelian. Then $H = \langle x_1, \ldots, x_d \rangle$ is finitely generated and Engelian, and, arguing as in Chapter 4, it follows that $\mathrm{gr}(H)$ is nilpotent. Assuming H were necessarily residually nilpotent, it would follow that H itself was nilpotent, and hence so was L. The following is thus an important open question:

Problem 9. *Are compact Hausdorff Engelian Lie rings necessarily residually nilpotent?*

From Chapter 5 we have:

Problem 10. *Does a finitely generated pro-p Lie ring L have finite Prüfer rank if and only if it does not involve* $W_\infty(\mathbb{F}_p) = \varprojlim_{n\geq 1} W_n(\mathbb{F}_p)$ *as a closed section?*

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